Over View of Control Charts

1·2 CONTROL CHARTS

The epoch-making discovery and development of control charts was made by a young physicist Dr. Walter A. Shewart of Bell Telephone Laboratories in 1924 and the following years. Based on the theory of probability and sampling Shewhart's control charts provide a powerful tool of discovering and correcting the assignable causes of variation outside the "stable pattern" of chance causes, thus enabling us to stabilize and control our processes at desired performances and thus bring the process under statistical control.

In industry one is faced with two kinds of problems : (*i*) to check whether the process is conforming to standards laid down, and (*ii*) to improve the level of standard and reduce variability consistent with cost considerations. Shewhart's control charts provide an answer to both. Control chart, as conceived and devised by Shewhart, is a simple pictorial device for detecting unnatural patterns of variations in data resulting from repetitive processes *i.e.*, control charts provide criteria for detecting lack of statistical control. Control charts are simple to construct and easy to interpret and tell us at a glance whether the sample point falls within 3-σ control limits (discussed below) or not. Any sample point going outside the 3-σ control limits is an indication of the lack of statistical control, *i.e.*, presence of some assignable causes of variation which must be traced, identified and eliminated.

A typical control chart consists of the following three horizontal lines :

(*i*) A central line (*CL*) to indicate the desired standard or the level of the process.

(*ii*) Upper Control Limit (*UCL*), and

(*iii*) Lower Control Limit (*LCL*),

together with a number of sample points as exhibited in the following diagram which depicts the principle of Shewhart's control chart.

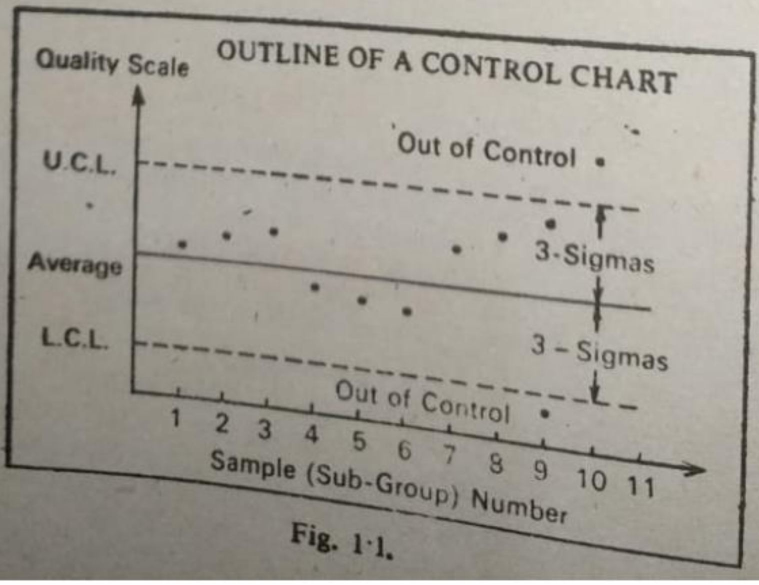


Fig. 1·1.

In the control chart, Upper Control Limit (*UCL*) and Lower Control Limit (*LCL*) are usually plotted as dotted lines and central line (*CL*) is plotted as a bold (dark) line. If t is the underlying statistic then these values depend on the sampling distribution of t and are given by

$$UCL = E(t) + 3\ \text{S.E.}(t)$$
$$LCL = E(t) - 3\ \text{S.E.}(t)$$
$$CL = E(t)$$

1·3. 3-σ CONTROL LIMITS

3-σ limits were proposed by Dr. Shewhart for his control charts from various considerations, the main being probabilistic considerations. Consider the statistic $t = t(x_1, x_2, \ldots, x_n)$, a function of the sample observations $x_1, x_2, \ldots, x_n$. Let

$$E(t) = \mu_t \text{ and } \text{Var}(t) = \sigma_t^2$$

If the statistic t is normally distributed, then from the fundamental area property of the normal distribution, we have

$$P[\mu_t - 3\sigma_t < t < \mu_t + 3\sigma_t] = 0{\cdot}9973$$
$$\Rightarrow \quad P[\,|\,t - \mu_t\,|\, < 3\sigma_t] = 0{\cdot}9973$$
$$\Rightarrow \quad P[\,|\,t - \mu_t\,|\, > 3\sigma_t] = 0{\cdot}0027$$

In other words, the probability that a random value of t goes outside the $3-\sigma$ limits, *viz.*, $\mu_t \pm 3\sigma_t$ is 0·0027, which is very small. Hence, if t is normally distributed, the limits of variation should be between $\mu_t + 3\sigma_t$ and $\mu_t - 3\sigma_t$ which are termed respectively the *Upper Control Limit* (*UCL*) and *Lower Control Limit* (*LCL*). If, for the ith sample, the observed t_i lies between the upper and lower control limits, there is nothing to worry as in such a case variation between samples is attributed to chance *i.e.*, in this case the process is in statistical control. It is only when any observed t_i falls outside the control limits, it is considered to be a danger signal indicating that some assignable cause has crept in which must be identified and eliminated.

Remarks 1. If the assumption regarding normality of the statistic t does not hold, then the above argument does not remain strictly valid. In practice, the quality characteristic can seldom be supposed to be exactly normal. For non-normal population, (i.e., if the sampling distribution of statistic t is not normal) we apply *Chebychev's Inequality* in probability theory which states that for any constant $k > 0$,

$$P[\,|\,t - E(t)\,|\, < k] \geqslant 1 - \frac{\text{Var}(t)}{k^2}$$

$$\Rightarrow \quad P[\,|\,t - \mu_t\,|\, < 3\sigma_t] \geqslant 1 - \frac{\sigma_t^2}{9\sigma_t^2} = \frac{8}{9}$$

which is also fairly high for practical purposes and the above argument holds, more or less. However, in practice, σ_t is not known and is estimated from the sample data and consequently Chebychev's inequality does not hold if σ_t is not known.

Moreover, according to the central limit theorem in probability, the statistics of observations drawn from non-normal populations will exhibit nearly normal behaviour. Of course, such behaviour will be more closely normal, the closer the underlying population is to a normal distribution, but the principle applies in any case.

Hence, even for non-normal population 3-σ limits are almost universally used, as they have been found to be most suitable empirically in the sense that the 3-σ control charts have been found to give excellent protection against both types of wrong actions we can take *viz.*, looking for trouble when there is none and not looking for trouble when there really is one.

2. If none of the sample points falls outside the control limits and if there is no evidence of non-random variation within the limits, it does not imply the absence of assignable causes altogether. All we can infer is that the hypothesis of random variations alone is reasonable one and from management point of view, looking for special assignable causes at this stage is unlikely to be profitable.

3. It has been emphasized strongly by Dr. Shewhart that a production process should not be adjudged in statistical control unless the random variation pattern persists for quite some time and for a sizable volume of output. More specifically he states :

"This potential state of economic control can be approached only as a statistical limit even after the assignable causes of variability have been detected and removed. Control of this kind cannot be reached in a day. It cannot be reached in the production of a product in which only a few pieces are manufactured. It can, however, be approached scientifically in a continuing mass production."

Usually, a process should be considered in statistical control if the pattern of random variation is exhibited by a sequence of not less than twenty-five samples, each of size four.

1·4 TOOLS FOR S.Q.C

The following four, separate but related techniques, are the most important statistical tools for data analysis in quality control of the manufactured products.

1. *Shewhart's Control Chart for Variables i.e.*, for a characteristic which can be measured quantitatively. Many quality characteristics of a product are measurable and can be expressed in specific units of measurement such as diameter of a screw, tensile strength of steel pipe, specific resistance of a wire, life of an electric bulb, etc. Such variables are of continuous type and are regarded to follow normal probability law. For quality control of such data, two types of control charts are used and technically these charts are known as :

(*a*) *Charts for $\bar{X}$ (mean) and R (Range),*

and (*b*) *Charts for $\bar{X}$ (mean) and σ (standard deviation).*

2. *Shewhart's Control Chart for fraction Defective or p-chart.* This chart is used if we are dealing with attributes in which case the quality characteristics of the product are not amenable to measurement but can be identified by their absence or presence from the product or by classifying the product as defective or non-defective.

3. *Shewhart's Control Chart for the 'Number of Defects" per unit or c-Chart.* This is usually used with advantage when the characteristic

representing the quality of a product is a discrete variable *e.g.*, (*i*) the number of defective rivets in an aircraft wing and (*ii*) the number of surface defects observed in a roll of coated paper or a sheet of photographic film.

4. The portion of the sampling theory which deals with the quality protection given by any specified sampling acceptance procedure.

1·5 CONTROL CHARTS FOR VARIABLES

These charts may be applied to any quality characteristic that is measurable. In order to control a measurable characteristic we have to exercise control on the measure of location as well as the measure of dispersion. Usually $\bar{X}$ and R charts are employed to control the mean (location) and standard deviation (dispersion) respectively of the characteristic.

1·5·1. $\bar{X}$ and R Charts. No production process is perfect enough to produce all the items exactly alike. Some amount of variation, in the produced items, is inherent in any production scheme. This variation is the totality of numerous characteristics of the production process *viz.*, raw material, machine setting and handling, operators, etc. As pointed out earlier, this variation is the result of (*i*) chance causes and (*ii*) assignable causes. The control limits in the $\bar{X}$ and R charts are so placed that they reveal the presence or absence of assignable causes of variation in the

(*a*) average—mostly related to machine setting, and

(*b*) range—mostly related to negligence on the part of the operator.

Steps for $\bar{X}$ and R Charts :

1. *Measurement.* Actually the work of a control chart starts first with measurements. Any method of measurement has its own inherent variability. Errors in measurement can enter into the data by :

(*i*) the use of faulty instruments,

(*ii*) lack of clear-cut definitions of quality characteristics and the method of taking measurements, and

(*iii*) lack of experience in the handling or use of the instrument, etc.

Since the conclusions drawn from control chart are broadly based on the variability in the measurements as well as the variability in the quality being measured, it is important that the mistakes in reading measurement instruments or errors in recording data should be minimised so as to draw valid conclusions from control charts.

2. *Selection of Samples or Sub-groups.* In order to make the control chart analysis effective, it is essential to pay due regard to the rational selection of the samples or sub-groups. The choice of the sample size n and the frequency of sampling *i.e.*, the time between the selection of two groups, depend upon the process and no hard and fast rules can be laid down for this purpose. Usually n is taken to be 4 or 5 while the frequency of sampling depends on the state of the control exercised. Initially more frequent samples will be required (15 to 30 minutes) and once a state of control is maintained, the frequency may be relaxed. Normally 25 samples of size 4 each or 20 samples of size 5

each under control will give good estimate of the process average and dispersion.

Remark. While collecting data it may not be necessary to go exactly at the specified time, in fact this should not be practised. This is to avoid (*i*) the operative being careful at the time of sampling, or (*ii*) any periodicities of the process to coincide with sampling.

3. *Calculation of $\bar{X}$ and R for each Sub-group.* Let X_{ij}, $j=1,2,\ldots,n$ be the measurements on the ith sample $(i=1,2,\ldots,k)$. The mean $\bar{X}_i$, the range R_i and the standard deviation s_i for the ith sample are given by

$$\left.\begin{aligned} \bar{X}_i &= \frac{1}{n}\sum_j X_{ij} \\ R_i &= \max_j X_{ij} - \min_j X_{ij} \\ s_i^2 &= \frac{1}{n}\sum_j (X_{ij}-\bar{X}_i)^2 \end{aligned}\right\} \quad (i=1,2,\ldots,k) \quad (1\cdot1)$$

Next we find $\bar{\bar{X}}$, $\bar{R}$ and $\bar{s}$, the averages of sample means, sample ranges and sample standard deviations, respectivly, as follows :

$$\left.\begin{aligned} \bar{\bar{X}} &= \frac{1}{k}\sum_i \bar{X}_i \\ \bar{R} &= \frac{1}{k}\sum_i R_i \\ \bar{s} &= \frac{1}{k}\sum_i s_i \end{aligned}\right\} \quad \ldots(1\cdot2)$$

4. *Setting of Control Limits.* It is well known that if σ is the process standard deviation (standard deviation of the universe from which samples are taken), then the standard error of sample mean is $\sigma/\sqrt{n}$, where n is the sample size *i.e.*,

$$S.E.\,(\bar{X}_i)=\sigma/\sqrt{n},\ (i=1,2,\ldots,k)$$

Also from the sampling distribution of range we know that

$$E(R)=d_2.\sigma$$

where d_2 is a constant depending on the sample size. Thus an estimate of σ can be obtained from $\bar{R}$ by the relation :

$$\bar{R}=d_2.\,\sigma \Rightarrow \sigma=\bar{R}/d_2 \quad \ldots(1\cdot3)$$

Also $\bar{\bar{X}}$ gives an unbiased estimate of the population mean μ, since

$$E(\bar{\bar{X}})=\frac{1}{k}\sum_{i=1}^{k} E(\bar{X}_i)=\mu$$

Control Limits for $\bar{X}$-chart :

Case 1. *When standards are given i.e.,* both μ and σ are known. The 3-σ control limits for $\bar{X}$ chart are given by

$$E(\bar{X})\pm 3\ S.E.\,(\bar{X})=\mu\pm 3\sigma/\sqrt{n}=\mu\pm A\sigma,\ (A=3/\sqrt{n})$$

If μ' and σ' are known or specified values of μ and σ respectively, then

$$\left.\begin{aligned} UCL_{\bar{X}} &= \mu' + A\sigma' \\ LCL_{\bar{X}} &= \mu' - A\sigma' \end{aligned}\right] \quad \ldots(1\cdot4)$$

Case 2. *Standards not given.* If both μ and σ are unknown, then using their estimates $\bar{\bar{X}}$ and σ given in (1·2) and (1.3) respectively, we get the 3-σ control limits on the $\bar{X}$ chart as :

$$\bar{\bar{X}} \pm 3\frac{\bar{R}}{d_2}\cdot\frac{1}{\sqrt{n}} = \bar{\bar{X}} \pm \left(\frac{3}{d_2\sqrt{n}}\right)\bar{R} = \bar{\bar{X}} \pm A_2\bar{R}, \left(A_2 = \frac{3}{d_2\sqrt{n}}\right)$$

$$\therefore \quad \left.\begin{aligned} UCL_{\bar{x}} &= \bar{\bar{X}} + A_2\bar{R} \\ LCL_{\bar{x}} &= \bar{\bar{X}} - A_2\bar{R} \end{aligned}\right\} \quad \ldots(1.4a)$$

Since d_2 is a constant depending on n, $A_2 = 3/(d_2\sqrt{n})$ also depends only on n and its values have been computed and tabulated for different values of n from 2 to 25 and are given in Table *VIII* in the appendix at the end of the book.

If on the other hand, the control limits are to be obtained in terms of $\bar{s}$ rather than $\bar{R}$, then an estimate of σ can be obtained from the relation

$$\bar{s} = C_2\sigma \Rightarrow \sigma = \bar{s}/C_2 \quad \ldots(1\cdot4b)$$

where

$$C_2 = \sqrt{\frac{2}{n}} \cdot \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!}$$

$$\therefore \quad \left.\begin{aligned} UCL_{\bar{x}} &= \bar{\bar{X}} + \left(\frac{3}{\sqrt{n}C_2}\right)\bar{s} = \bar{\bar{X}} + A_1\bar{s} \\ LCL_{\bar{x}} &= \bar{\bar{X}} - \left(\frac{3}{\sqrt{n}C_2}\right)\bar{s} = \bar{\bar{X}} - A_1\bar{s} \end{aligned}\right\} \quad \ldots(1\cdot4c)$$

The factor $A_1 = 3/(\sqrt{n}C_2)$ has been tabulated for different values of n from 2 to 25.

Control Limits for R-chart. 3-σ control limits for R-chart are given by $E(R) \pm 3\sigma_R$. $E(R)$ is estimated by $\bar{R}$ and σ_R is estimated from the relation

$$\sigma_R = cE(R) = c\bar{R}$$

where c is constant depending on n.

$$\therefore \quad \left.\begin{aligned} UCL_R &= \bar{R} + 3c\bar{R} = (1+3c)\bar{R} = D_4\bar{R} \\ LCL_R &= \bar{R} - 3c\bar{R} = (1-3c)\bar{R} = D_3\bar{R} \end{aligned}\right\} \quad \ldots(1\cdot5)$$

The values of $D_4 = 1+3c$ and $D_3 = 1-3c$ have been tabulated for different values of n from 2 to 25.

However, the control limits for R-chart can be obtained directly from the assumed or known value of σ as follows :

$$\left.\begin{array}{l} UCL_R = D_2\sigma \\ UCL_R = D_1\sigma \end{array}\right\} \quad \ldots(1\cdot 5a)$$

where D_1 and D_2 are tabulated for values of n from 2 to 25.

Remark. It should be noted carefully that the control limits for $\bar{X}$ and R charts are based upon the assumption that different samples or sub-groups are of constant size n.

5. **Construction of Control Chart for $\bar{X}$ and R** *i.e., plotting of Central Line and the Contral Limits.* Control charts are plotted on a rectangular co-ordinate axis—vertical scale (ordinate) representing the statistical measures $\bar{X}$ and R, and horizontal scale (abscissa) representing the sample number. Hours, dates or lot numbers may also be repesented on the horizontal scale. Sample points mean or range are indicated on the chart by points, which may or may not be joined.

For $\bar{X}$-chart, the central line is drawn as a *solid* horizontal line at $\bar{\bar{X}}$ and $UCL_{\bar{x}}$ and $LCL_{\bar{x}}$ are drawn at the computed values as *dotted* horizontal lines.

For R chart, the central line is drawn as a solid horizontal line at $\bar{R}$ and UCL_R is drawn at the computed values as a dotted horizontal line. If the sample size is seven or more ($n \geqslant 7$), LCL_R is drawn as dotted horizontal line at the computed value otherwise ($n < 7$) LCL_R is taken as zero.

Remarks on $\bar{X}$ and R-Charts. We give below some of the very important remarks which should be clearly understood by the reader.

1. The values of constants $A, A_1, A_2, D_1, D_2, D_3$, and D_4, for different values of n are given in the table VIII in the Appendix at the end of the book.

2. $\bar{X}$-chart reveals undesirable variations between samples as far as their averages are concerned while the R-chart reveals any undesirable variation within samples.

3. For a process to be working under statistical control, points both in the $\bar{X}$ and R charts should lie between the control limits. A process which is not in statistical quality control suggests the presence of assignable causes of variation which throw the process out of control. These causes must be traced and eliminated so that the process may return to operation under stable statistical conditions. Reasons for the process being out of control vary from faulty tools, a sudden significant change in properties of new materials in a new consignment, breakdown of the lubrication system, faults in timing or speed mechanisms, etc. Tracing these causes is sometimes simple and straightforward, but in some cases it may be a rather lengthy and complicated business, especially when the process is subject to the combined effect of several external causes simultaneously.

If the process is found to be in statistical control, a comparison between the required specifications and the process capability may be carried out to determine whether the two are compatible. Should the specified tolerances prove to be too tight for process capability, there are three possible alternatives :

(*i*) Re-evaluate the specifications : Are the tight tolerances really necessary for effective performance, or could they, perhaps, be relaxed with no detriment to the quality of the product.

(*ii*) If relaxation of the specifications is not acceptable, perhaps a more accurate process should be selected for the purpose ?

(*iii*) If both the previous alternatives are out of the question, a 100 per cent inspection must be undertaken to sort out the defective products.

4. If all the points in both the charts remain within trial limits, then these limits are accepted as final, and used for maintaining control charts for subsequent production. If, however, some of the points go outside the limits in one of the charts then it is concluded that these samples were produced when the process was not in control and these samples are rejected as unusable. Then a second set of trial limits is constructed, using only the remaining samples, and using these fresh control limits, new charts are constructed and the remaining samples are plotted on the new charts. If all the sample points now remain within the new control limits, they are accepted as final otherwise the same procedure as described above is followed to get a third set of trial control limits. The control limits are accepted as final only when all the sample points on which they are based remain within these limits.

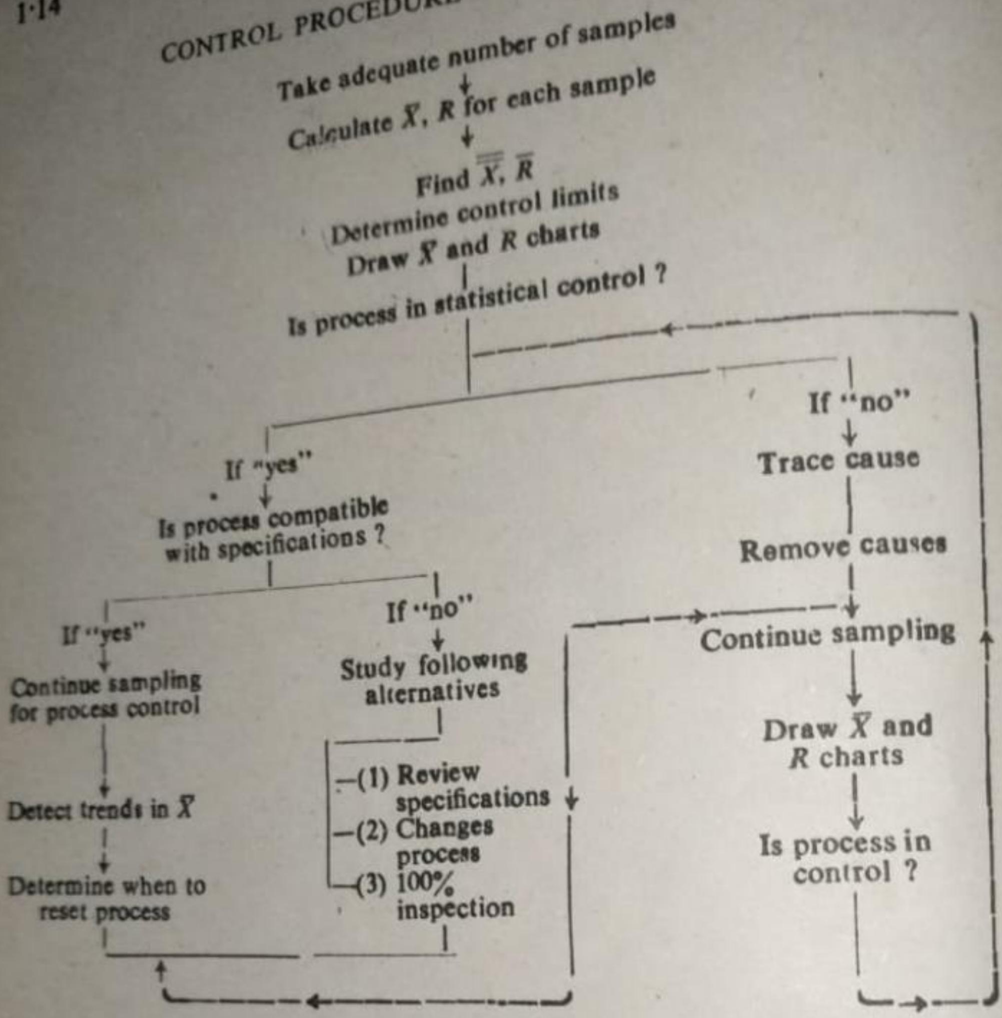
5. Should the routine results show a better degree of uniformity than that expected from the standard there is evidence that the acceptance standard is too loose. The latest data must then be used to re-estimate current standard quality, which is then used for future control. This procedure should also be adopted at regular intervals, thereby producing a gradual improvment in quality standards.

6. We give the on page 1·14 diagrammatic representation of the control procedure associated with $\bar{X}$ and R charts.

1·5·2. Criterion for Detecting Lack of Control in $\bar{X}$ and R Charts. As pointed out earlier, the main object of the control chart is to indicate when a process is not in control. The crtieria for detecting lack of control are, therefore, of fundamental and crucial importance. The pattern of the sample points in a control chart is the key to the proper interpretation of the working of the process. The following situations depict lack of control :

1. *A point outside the control limits*. The probabilistic considerations provide a basis for hunting for lack of control in such a situation A point going outside control limits is a clear indication of the presence of assignable causes of variation which must be searched and corrected. A point outside the control limit may result from an increased dispersion or change in level. Lack of uniformity may be due to the variation in the quality of raw materials, deficiency in the skill of the operators, loss of

CONTROL PROCEDURE FOR $\bar{X}$ AND R-CHARTS



alignment among machines, change of working conditions, etc. It may be indicated by a point (or points) above the upper control limit for ranges. It may also result in points outside the control limits for means.

2. *A run of seven or more points.* Although all the sample points are within control limits usually the pattern of points in the chart indicates assignable causes. One such situation is a run of 7 or more points above or below the central line in the control chart. Such runs indicate shift in the process level. On R-chart a run of points above the central line is indicative of increase in process spread and therefore represents an undesirable situation, while a run below the central line indicates an improvement in the sense that the variability has been reduced *i.e.*, the process could hold to a closer tolerance.

3. One or more points in the vicinity of control limits or a run of points beyond some secondary limits *e.g.*, a run of 2, 3 points beyond 2-σ limits or a run of 4, 5 points beyond 1-σ limits.

4. The sample points on $\bar{X}$ and R charts, too close to the central line, exhibit another form of assignable-cause. This situation (Fig. 1·2) represents systematic differences within samples or sub-groups and results from improper selection of samples and biases in measurements.

SAMPLE POINTS TOO CLOSE TO THE CENTRAL LINE

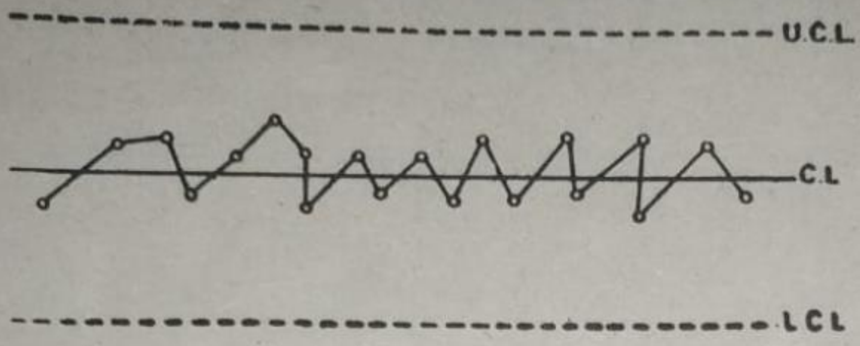


Fig. 1·2.

5. *Presence of Trends*. The trends exhibited by sample points on the control chart are also an indication of assignable cause. Trend pattern [Fig. 1·3(*a*) and 1·3(*b*)],a phenomenon usually observed in engineering industry, indicates the gradual shift in the process level. Trend may be upward or downward. Tools wear and the need for resetting machines often accounts for such a shift, and it is essential to determine when machine resetting becomes desirable bearing in mind that too frequent adjustments are a serious setback to production output.

PRESENCE OF TRENDS IN CONTROL CHART

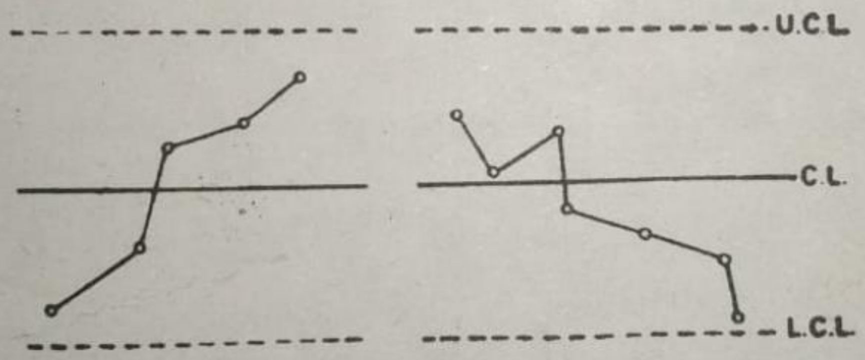


Fig. 1·3(*a*) Fig. 1·3(*b*)

6. *Presence of Cycles*. In some cases the cyclic pattern of points in the control chart (Fig. 1·4 on page 1·16) indicates the presence of assignable causes of variation. Such patterns are due to material or/and any mechanical reasons.

1·5·3. Interpretation of $\overline{X}$ and R charts. In order to judge if a process is in control, $\bar{X}$ and R charts should be examined together and the process should be deemed in statistical control if both the charts show a state of control. Situations exist where R-chart is in a state of control but $\bar{X}$ chart is not. We summarise below, in a tabular form, such different situations and the interpretation to be accorded to each.

PRESENCE OF CYCLES IN CONTROL CHARTS

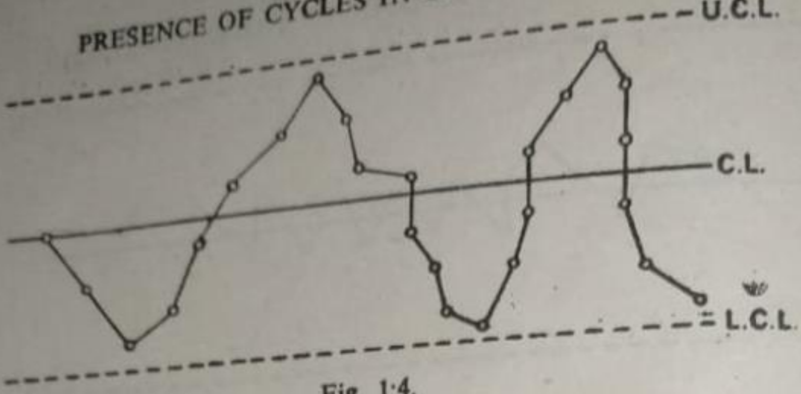


Fig. 1·4.

Sl. No.	Situation in R-chart	Situation in $\bar{X}$-chart	Interpretation
1	In control	Points beyond limits only on one side	Level of process has shifted
2	In control	Points beyond limits on both the sides	Level of process is changing in erratic manner—frequent adjustments
3	Out of control	Points beyond limits on both sides	Variability has increased
4	Out of control	Out of control on one side	Both level and variability have changed
5	In control	Run of 7 or more points on one side of central line	Shift in process level
6	In control	Trend of 7 or more points. No point outside control limits	Process level is gradually changing
7	Runs of 7 or more points above central line	— — —	Variability has increased
8	Points too close to the central line	— — —	Systematic differences within sub-groups
9	— — —	Points too close to the central line	Systematic differences within sub-groups

Example 1·1. *Construct a control chart for mean and the range for the following data on the basis of fuses, samples of 5 being taken every hour (each set of 5 has been arranged in ascending order of magnitude). Comment on whether the production seems to be under control, assuming that these are the first data.*

42	42	19	36	42	51	60	18	15	69	64	61
65	45	24	54	51	74	60	20	30	109	90	78
75	68	80	69	57	75	72	27	39	113	93	94
78	72	81	77	59	78	95	42	62	118	109	109
87	90	81	84	78	132	138	60	84	153	112	136

Solution. [*Madras Univ.B. Sc. (Main) 1982*]

Sample No. (1)	*Sample Observations* (2)					*Total* (3)	*Sample Mean* $\bar{X}$ (4)	*Sample Range R*
1	42	65	75	78	87	347	69·4	45
2	42	45	68	72	90	317	63·4	48
3	19	24	80	81	81	285	57·0	62
4	36	54	69	77	84	320	64·0	48
5	42	51	57	59	78	287	57·4	36
6	51	74	75	78	132	410	82·0	81
7	60	60	72	95	138	425	85·0	78
8	18	20	27	42	60	167	33·4	42
9	15	30	39	62	84	230	46·0	69
10	69	109	113	118	153	562	112·4	84
11	64	90	93	109	112	468	93·6	48
12	61	78	94	109	136	478	95·6	75
						Total	859·2	716

From above data, we get

$$\bar{\bar{X}}=\frac{1}{12}\Sigma\,\bar{X}_i=\frac{859{\cdot}2}{12}=71{\cdot}60$$

$$\bar{R}=\frac{1}{12}\Sigma\,R_i=\frac{716}{12}=59{\cdot}67$$

From the tables, for $n=5$, we have

$$A_2=0{\cdot}58,\ D_3=0 \text{ and } D_4=2{\cdot}11$$

$\bar{X}$-Chart

$$UCL_{\bar{x}}=\bar{\bar{X}}+A_2\bar{R}$$

$$=71{\cdot}6+0{\cdot}58\times59{\cdot}67=106{\cdot}21$$

$$LCL_{\bar{x}}=\bar{\bar{X}}-A_2\bar{R}$$

$$=71{\cdot}60-0{\cdot}58\times59{\cdot}67=36{\cdot}99$$

$$CL_{\bar{x}}=\bar{\bar{X}}=71{\cdot}60$$

From the chart, on page 1·18 clearly the process is out of control since the points coresponding to 8th and 10th samples lie outside the control limits.

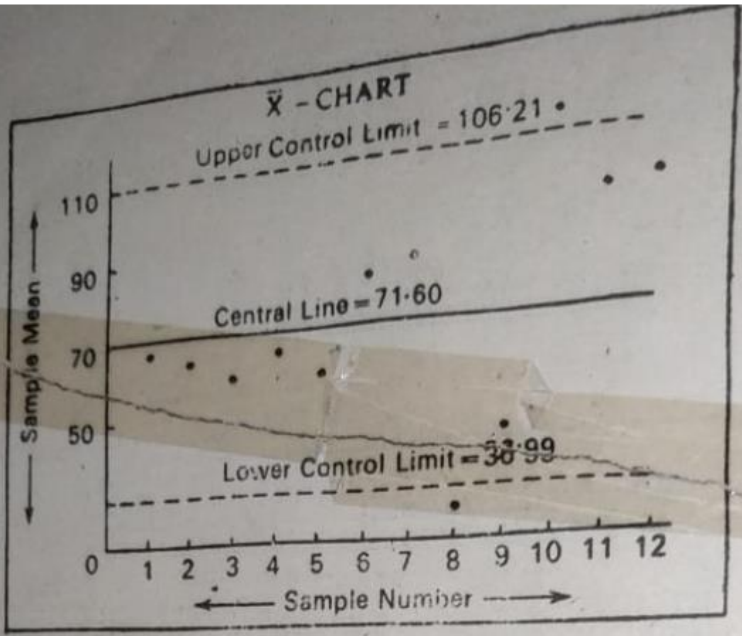


Fig. 1·5.

R-Chart ;

$$LCL_R = 0$$
$$UCL_R = D_4\overline{R} = 2{\cdot}11 \times 59{\cdot}67 = 125{\cdot}904$$
$$CL_R = 59{\cdot}67$$

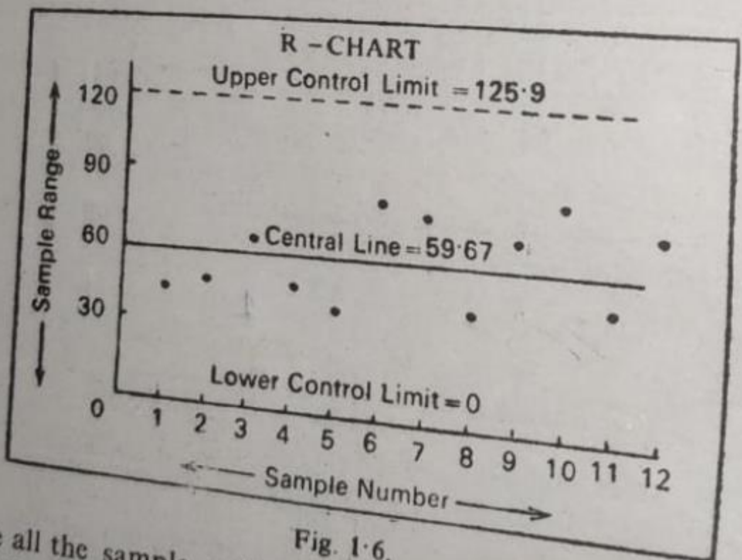


Fig. 1·6.

Since all the sample points fall within control limits, this chart shows that process is in control.

Although R-chart depicts control, the process can't be regarded to be in statistical control since $\overline{X}$-chart shows lack of control (See remark 3 page 1·12).

The choice between $\overline{X}$ and R chart is a managerial problem. It is better to construct R-chart first. If the R-chart indicates that the dispersion of the quality by the process is out of control, generally it is better not to construct $\overline{X}$-chart, until the quality dispersion is brought under control.

Example 1·2. *The following are the mean and ranges of 20 samples of size 5 each. The data pertain to the overall length of a fragementation bomb base manufactured during the war by the American store camp.*

Group No.	Mean	Range	Group No.	Mean	Range
1	0·8372	0·010	11	0·8380	0·006
2	0·8324	0·009	12	0·8322	0·002
3	0·8318	0·008	13	0·8356	0·013
4	0·8344	0·004	14	0·8322	0·005
5	0·8346	0·005	15	0·8404	0·008
6	0·8332	0·011	16	0·8372	0·011
7	0·8340	0·009	17	0·8282	0·006
8	0·8344	0·003	18	0·8346	0·006
9	0·8308	0·002	19	0·8360	0·004
10	0·8350	0·006	20	0·8374	0·006

(a) From these data, obtain the control limits for $\bar{X}$ and R charts to control the length of bomb bases produced in the future.

(b) The above samples were taken every 15 minutes in order of production. The production rate was 350 units per hour and the tolerances were 0·820 and 0·840 inches.

On the assumption that the lengths of the bomb bases are normally distributed, what percentage of the bomb base would you estimate to have length outside the tolerance limits when the process is under control at the levels indicated by the above data ?

Solution. (*a*) $$\bar{\bar{X}}=\frac{1}{20}\sum_{i=1}^{20}\bar{X}_i=\frac{16\cdot6796}{20}=0\cdot83398$$

$$\bar{R}=\frac{1}{20}\sum_{i=1}^{20}R_i=\frac{0\cdot133}{20}=0\cdot00665$$

From tables for $n=5, A_2=0\cdot58$, $D_3=0$ and $D_4=2\cdot12$

$$UCL_{\bar{x}}=\bar{\bar{X}}+A_2\bar{R}=0\cdot83398+0\cdot58\times0\cdot00665=0\cdot837837$$

$$CL_{\bar{x}}=0\cdot83398$$

$$LCL_{\bar{x}}=\bar{\bar{X}}=A_2\bar{R}-0\cdot83398-0\cdot58\times0\cdot00665=0\cdot830123$$

We see that values of $\bar{X}$ corresponding to the sub-groups 11 and 17 are outside the control limits. So excluding these values, we get

$$\bar{\bar{X}}'=\frac{15\cdot0134}{18}=0\cdot834077,\ \bar{R}'=\frac{0\cdot121}{18}=0\cdot00672$$

So the new control limits of $\bar{X}$ chart are

$$\bar{\bar{X}}'\pm A_2\bar{R}'=0\cdot834077\pm0\cdot58\times0\cdot00672=0\cdot837975 \text{ and } 0\cdot831179$$

which may be regarded as the final limits of $\bar{X}$, since no value except the rejected ones is outside the limits.

$$UCL_R=D_4\bar{R}=2\cdot12\times0\cdot00665=0\cdot014088$$

$$CL_R=0\cdot00665$$

$$LCL_R=0$$

From the given values of R we see that the values corresponding to all the sub-groups fall inside the control limits, which may be taken as the final control limits.

(*b*) We are given that

Upper Tolerance Limit (U.T.L.) $= 0{\cdot}840''$

Lower Tolerance Limit (L.T.L.) $= 0{\cdot}820''$

Assuming that the process is in control the, random variable $\overline{X}$ (the length of the bomb bases) is normally distributed with mean and *s.d.* given by

$$\hat{\mu} = \overline{\overline{X}}' = 0{\cdot}834077 \text{ and } \hat{\sigma} = \overline{R}'/d_2 = \frac{0{\cdot}00672}{2{\cdot}326} = 0{\cdot}002889$$

Hence the proportion 'p' of defectives when the process is in control is given by

$$p = P[X \text{ lies outside tolerance limits}]$$
$$= 1 - P[0{\cdot}82 < X < 0{\cdot}84]$$
$$= 1 - P[-4{\cdot}7982 < Z < 4{\cdot}999], \text{ where } Z = \frac{X - \hat{\mu}}{\hat{\sigma}}$$
$$= 0{\cdot}0201825 \text{ (From Biometrika Tables)}$$

Hence the per cent fraction defective when the process is in control is

$$100 \times 0{\cdot}0201825 = 2{\cdot}01825$$

Example 1·3. (*a*) *Show that p_n, the probability of the mean of a random sample of size n exceeding UCL* $= \mu' + 3\sigma'/\sqrt{n}$ *when the population mean has shifted to* $\mu' + k\sigma'$ *is* $G(3 - k\sqrt{n})$, *where*

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}u^2}\, du$$

[*Delhi Univ. B. Sc.* (*Hons. Stat.*), *1983*]

(*b*) *If the rth sample mean is the first to exceed UCL, shows that* $E(r) = 1/p_n$.

[*Gauhati Univ. B. Sc.* (*Hons. Stat.*), *1983*]

Solution. (*a*) If $\overline{X}$ is the mean of a sample of size n from a population with mean μ' and standard deviation σ' then, $\overline{X}$ is normally distributed if population is normal and $\overline{X}$ is asymptotically normally distributed (by Central Limit Theorem) is population is not normal, with mean μ' and standard deviation $\sigma'/\sqrt{n}$. After the shift of the mean to $\mu' + k\sigma'$ has taken place, $\overline{X} \sim N(\mu' + k\sigma', \sigma'^2/n)$

$$\therefore \quad p(\bar{x}) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma'} \cdot \exp\left[-\frac{n}{2\sigma'^2}\left(\bar{x} - \mu' - k\sigma'\right)^2\right], \; -\infty < \bar{x}' < \infty$$

$$\therefore \quad p_n = P(\overline{X} > UCL) = \int_{UCL}^{\infty} p(\bar{x})\, d\bar{x}$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma'} \int_{\mu' + 3\sigma'/\sqrt{n}}^{\infty} \exp\left[-\frac{n}{2\sigma'^2}\left(\bar{x} - \mu' - k\sigma'\right)^2\right] d\bar{x}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{(3-k\sqrt{n})}^{\infty}\exp\left(-\tfrac{1}{2}t^2\right)dt,\quad\left(t=\frac{\bar{x}-\mu'-k\sigma'}{\sigma'/\sqrt{n}}\right)$$

$$=G(3-k\sqrt{n})\qquad\text{[By definition of } G(.)]$$

(*b*) If $\bar{X}_i$ is the mean of the *i*th sample of size *n* then

$$p_n=P\{\bar{X}_i>UCL\}$$
$$q_n=P\{\bar{X}_i\leqslant UCL\}=1-p_n$$

If *r*th sample mean is the first to exceed the *UCL*, the preceding $(r-1)$ sample means must be $\leqslant UCL$. Thus if Y is the random variable such that $Y=r$ (1, 2, ...) implies that the *r*th sample mean is the first to exceed *UCL* then Y follows the geometric distribution with probability function

$$f(r)=P(Y=r)=q_n^{r-1}.p_n\qquad(r=1,2,\ldots)$$

$$\therefore\quad E(Y)=\sum_{r=1}^{\infty}r\,f(r)=\sum rq_n^{r-1}.p_n=p_n[1+2q_n+3q_n^2+\ldots]$$

Let
$$S=1+2q_n+3q_n^2+4q_n^3+\ldots$$
$$\therefore\quad q_n.S=\quad q_n+2q_n^2+3q_n^3+\ldots$$
$$(1-q_n)S=1+q_n+q_n^2\ldots=\frac{1}{(1-q_n)}$$
$$\Rightarrow\quad S=1/(1-q_n)^2$$
$$\therefore\quad E(Y)=p_n/(1-q_n)^2=1/p_n$$

Example 1·4. *Show that the probability that at least one of the two points* $\bar{X}$ *and R goes outside the control limits is*

$$1-\left[G(\sqrt{n}T-3\rho)-G(\sqrt{n}T+3\rho)\right]\left[P\left(\frac{R}{\sigma}\leqslant D_2\rho\right)-P\left(\frac{R}{\sigma}\leqslant D_1\rho\right)\right]$$

where $\rho=\sigma'/\sigma$, $T=(\mu'-\mu)/\sigma$ and $G(x)=\frac{1}{\sqrt{2\pi}}\int_x^{\infty}e^{-\frac{1}{2}t^2}dt,$

assuming that the control charts are based on μ' *as population mean and* σ' *as population standard deviation, where the actual values of these parameters are* μ *and* σ *respectively.* [*Delhi Univ. M. Sc. 1982*]

Solution. Probability (p_1) that all the points on the *R*-charts lie within control limits is given by

$$p_1=P[D_1\sigma'\leqslant R\leqslant D_2\sigma']=P(R\leqslant D_2\sigma')-P(R\leqslant D_1\sigma)$$
$$P=\left(\frac{R}{\sigma}\leqslant D_2\frac{\sigma'}{\sigma}\right)-P\left(\frac{R}{\sigma}\leqslant D_1\frac{\sigma'}{\sigma}\right),\ \sigma>0$$
$$=P\left(\frac{R}{\sigma}\leqslant D_2\rho\right)-P\left(\frac{R}{\sigma}\leqslant D_1\rho\right)\qquad\ldots(*)$$

Probability (p_2) that all the points on $\bar{X}$-chart lie within the control limits is

VIII—FACTORS USEFUL IN THE CONSTRUCTION OF CONTROL CHARTS

Sample size	Mean chart			Standard deviation chart					Range chart				
	Factors for control limits			Factors for central line	Factors for control limits				Factors for central line	Factors for control limits			
n	A	A_1	A_2	c_2	B_1	B_2	B_3	B_4	d_2	D_1	D_2	D_3	D_4
2	2·121	3·760	1·880	0·5642	0	1·843	0	3·267	1·128	0	3·686	0	3·267
3	1·732	2·394	1·023	0·7236	0	1·858	0	2·568	1·693	0	4·358	0	2·575
4	1·500	1·880	0·729	0·7979	0	1·808	0	2·266	2·059	0	4·698	0	2·282
5	1·342	1·596	0·577	0·8407	0	1·756	0	2·089	2·326	0	4·918	0	2·115
6	1·225	1·410	0·483	0·8686	0·026	1·711	0·030	1·970	2·534	0	5·078	0	2·004
7	1·134	1·277	0·419	0·8882	0·105	1·672	0·118	1·882	2·704	0·205	5·203	0·076	1·924
8	1·061	1·175	0·373	0·9027	0·167	1·638	0·185	1·815	2·847	0·387	5.307	0·136	1·864
9	1·000	1·094	0·337	0·9139	0·219	1·609	0·239	1·761	2·970	0·546	5·394	0·184	1·816
10	0·949	1·028	0·308	0·9227	0·262	1·584	0·284	1·716	3·078	0·687	5·469	0·223	1·777
11	0·905	0·973	0·285	0·9300	0·299	1·561	0·321	1·679	3·173	0·812	5·534	0·256	1·744
12	0·866	0·925	0·266	0·9359	0·331	1·541	0·354	1·646	3·258	0·924	5·592	0·284	1·716
13	0·832	0·884	0·249	0·9410	0·359	1·523	0·382	1·618	3·336	1·026	5·646	0·308	1·692
14	0·802	0·848	0·235	0·9453	0·384	1·507	0·406	1·594	3·407	1·121	5·693	0·329	1·671
15	0·775	0·816	0·223	0·9490	0·406	1·492	0·428	1·572	3·472	1·207	5·737	0·348	1·652
16	[illegible]750	0·788	0·212	0·9523	0·427	1·478	0·448	1·552	3·532	1·285	5·779	0·364	1·636
17	0·728	0·762	0·203	0·9551	0·445	1·465	0·466	1·534	3·588	1·359	5·817	0·379	1·621
18	0·707	0·738	0·194	0·9576	0·461	1·454	0·482	1·518	3·640	1·426	5·854	0·392	1·608
19	0·688	0·717	0·187	0·9599	0·477	1·443	0·497	1·503	3·689	1·490	5·888	0·404	1·596
20	0·671	0·697	0·180	9·9619	0·491	1·433	0·510	1·490	3·735	1·548	5·922	0·414	1·586
21	0·655	0.679	0·173	0·9638	0·504	1·424	0·523	1·477	3·778	1·606	5·950	0·425	1·575
22	0·640	0·662	0·167	0·9655	0·516	1·415	0·534	1·466	3·819	1·659	5·979	0·434	1·566
23	0·626	0·647	0·162	0·9670	0·527	1·407	0·545	1·455	3·858	1·710	6·006	0·443	1·557
24	0·612	0·632	0·157	0·9684	0·538	1·399	0·555	1·445	3·895	1·759	6·031	0·452	1·548
25	0·600	0·619	0·153	0·9696	0·548	1·392	0·565	1·435	3·931	1·804	6·058	0·459	1·541