I2) **List Model**- consider n elements $e_1, e_2, \ldots e_n$ that are initially arranged in some ordered list. At each unit of time a request is made for one of 3 elements. e_i being requested independently of the part of probability P_i. After being request the element is then moved to the front of the list. i.e for instance if the present ordering is e_1, e_2, e_3, e_4 & e_3 is requested then the next ordering is e_3, e_1, e_2, e_4. To compute the expected partition of the element requested we start by conditioning on which element is selected.

SECTION-A

This yields E[Partition of element requested) $= \sum_{i=1}^{n} E$(position | e_i is selected) P_i

$= \sum_{i=1}^{n} E$[Position of e_i | e_i is selected] P_i

$= \sum_{i=1}^{n} E$[position of e_i] P_i

where the final equality used that the position of e_i & the event that e_i is selected are independent because, regardless of its position e_i is selected with probability P_i

Now position of $e_i = 1 + \sum I_j$ where $I_i = \begin{cases} 1 ; & \text{if } e_j \text{ preceeds } e_i \\ 0 ; & \text{ow} \end{cases}$

position of $E_i = 1 + \sum_{j \neq 1} I_j \Rightarrow E$[position of E_i] $= 1 + \sum_{j \neq 1} E(I_j)$

$= 1 + \sum_{j \neq 1} P(E_j \text{ preceedes } E_i)$ — (2)

we observe that E_j will preceede E_i if the most recent request of either of them is E_j, But given that a request is for either E_i or E_j The probability that it is for E_j is $P[E_j | E_i \text{ or } E_j] = P_j / P_i + P_j$

$\Rightarrow P[E_j \text{ Preceeds } E_i] = P_j / P_i + P_j$

$\Rightarrow E$[Position of E_i] $= 1 + \sum_{j \neq 1} P_j / P_i + P_j$ — (3)

from (1) & (3) E[position of element requested] $= 1 + \sum_{i=1}^{n} P_i$

$\sum P_j / P_i + P_j$

3) Uniqueness theorem

Statement - The 2 distribution f^n $F(x) = \{F_1(x), F_2(x)\}$ are identical iff their characteristic f^n $\phi_1(t)$ & $\phi_2(t)$ are identical.

Proof - let $F_1(x)$ & $F_2(x)$ be the 2 distribution function corresponding to a given characteristic fn ϕ_t then from the inversion theorem for any $a<b$ we have $F_1(b) - F_2(b) = F_1(a) - F_2(a)$ at all the common points of continuity of F_1 & F_2.

From the inversion theorem $F_2(b) - F_2(a) = F_1(b) - F_1(a) = c$

$F_1(b) - F_2(b) = F_1(a) - F_2(a) = c$.

let us allow b to be vary for a fixed a & WKT $F_1(\infty) = 1$, $F_2(\infty) =$

$F_1(\infty) - F_2(\infty) = 0$ then we have $F_1(b) - F_2(b) = F_1(a) - F_2(a) = c$

as $b \to \infty$

$F_1(a) - F_2(a) = 0 \Rightarrow F_1(a) = F_2(a)$

At all the continuity point of F_1 & F_2, $F_1 = F_2$

F is a uniquely determined by $\phi_X(t)$

Uses for uniqueness theorem

1) It is used in poisson's equation
2) It is used in division theorem
3) It is used in cauchy-kowalevski theorem
4) It is used for fundamental theorem of arithmetic
5) It is used in Holmgren's uniqueness theorem.

7) Kolmogorov's in equality

Statement - If $\{X_n, n\geq 1\}$ is a sequence of iid integrable random variable with common mean μ which is less than ∞ then then $\{X_n\}$ Obeys the SLLN. $\frac{S_n}{n} \xrightarrow{as} \mu$ as $n \to \infty$

Applications

1) If $X_1, X_2, \ldots X_n$ is a random sample drawn from population with the distribution f^n $F(x)$ where $x \in R$ then the empirical function defined as $G_n(x) = \left(\frac{\text{no of observations} \leq x}{n}\right) \ni$

$G_n(x) \xrightarrow{as} F(x)$ as $n \to \infty$

Proof - let us define $Y_k = \begin{cases} 1; & \text{if } X_k \leq x \\ 0; & \text{if } X_k > x \end{cases}$ where $k = 1, 2, \ldots n$

Y_k takes only 2 values either 0 or 1. Y_k are iid bernoulli rv & the

$P(Y_k = 1) = P(X_k \leq x) = F_k(x)$

$P(Y_k = 0) = P(X_k > x) = 1 - F_k(x)$ $\forall$ $k = 1, 2, \ldots n$

$\therefore$ $\{Y_n, n \geq 1\}$ Obeys SLLN from borel's SLLN

$\sum_{k=1}^{n} \frac{Y_k}{k} \xrightarrow{as} E\left(\frac{\sum Y_k}{k}\right)$ as $n \to \infty$

$\sum_{k=1}^{n} \frac{Y_k}{k} \xrightarrow{as}$ as $\Sigma\left(\frac{Y_k}{k}\right)$

ΣY_k represents how many X_k's $\leq x$

If $Y_k = 1 \Rightarrow \Sigma Y_k$'s $\leq x$ i.e. $\Sigma \frac{Y_k}{k}$ is nothing but $G_n(x) \xrightarrow{as} E\left(\frac{\Sigma Y_k}{k}\right)$

$E\left(\frac{\Sigma Y_k}{k}\right)$ as $n \to \infty$.

Hence the sequence of empirical distribution converges to the distribution function of a real valued rv at almost surely as $n \to \infty$.

8) Liapunov's form of CLT

Statement - let $x_1, x_2, \ldots x_n$ (or) $\{x_n, n \geq 1\}$ be the sequence of independent random variable then a positive no δ can be found such that as $n \to \infty$ $\frac{1}{B_n^{2+\delta}} \sum_{k=1}^{n} E\left(|X_k - \mu_k|^{2+\delta}\right) = \frac{V_{2+\delta}}{(B_n)^{2+\delta}} \to 0$

then $\{X_n\}$ satisfy CLT.

$$\lim_{n \to \infty} P\left[\frac{S_n - E(S_n)}{B_n} \leq x\right] \to F_z(x) \sim N(0,1)$$

Demoviere-Laplace CLT

Statement - let $\{X_n, n \geq 1\}$ be the sequence of independent random variable with the $P(X_k = 1) = p$. $P(X_k = 0) = 1-p$ the cummulative sum of square ie $S_n = \sum_{k=1}^{n} X_k$ is asymptotically normal $\frac{S_n - E(S_n)}{SD(S_n)} \xrightarrow{L} N(0,1)$ as $n \to \infty$.

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SECTION-B

10) i) If X_1 & X_2 are independent Poisson random variable with λ_1 & λ_2 then find $E[X_1 | X_1+X_2=n]$

Proof- Consider $P(X_1=k / X_1+X_2=n)$

$$= \frac{P(X_1=k, X_1+X_2=n)}{P(X_1+X_2=n)}$$

$$= \frac{P[X_1=k, X_2=n-k)}{P(X_1+X_2=n)}$$

$$= \frac{e^{-\lambda_1}\lambda_1^k}{k!} \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!} \Big/ \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= {}^nC_k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}; \; k=0,1,2,\dots n$$

which is binomial pmf with $P=\frac{\lambda_1}{\lambda_1+\lambda_2}$.

Hence $E[X_1 | X_1+X_2] = n \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$

ii) ST $E(\sum_{i=1}^{N} X_i] = E(n) E(X)$.

Proof- consider $E(\sum_{i=1}^{N} X_i) = E\left[E(\sum_{i=1}^{N} X_i / N)\right]$

$$= E\left[E(\sum_{i=1}^{n} X_i / N=n)\right]$$

$$= E\left[E(\sum_{i=1}^{n} X_i\right]$$

$$= E\left[\sum_{i=1}^{n} E(X_i)\right]$$

$= E[n \cdot E(X)]$ ($\because X_i$ & n_i are independent)

$E[n] E[X]$

15) Khintchines WLLN

Statement - Let $\{X_n; n \geq 1\}$ be a sequence of iid r.v.s. let $X_n = \frac{S_n - E(S_n)}{n}$ with CF $\phi_{X_n}(t)$ & with common mean $\mu < \infty$ then $X_n \xrightarrow{P} X$ as $n \to \infty$

Proof - Given $X_n = \frac{S_n - E(S_n)}{n}$

$$= \frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)}{n}$$

$$= \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} [E(X_i)]}{n}$$

$$= \frac{\sum_{i=1}^{n} X_i - n\mu}{n}$$

$$= \sum_{i=1}^{n} (X_i - \mu) / n$$

Consider $\phi_{X_n}(t) = E(e^{itX_n})$

$$= E\left(e^{it\left(\sum_{i=1}^{n} \frac{(X_i - \mu)}{n}\right)}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{i\left(\frac{t}{n}\right)(X_i - \mu)}\right]$$

$$= \prod_{i=1}^{n} E\left(e^{i(t/n)(X_i - \mu)}\right)$$

$$= \prod_{i=1}^{n} \phi_{(X_i - \mu)}\left(\frac{t}{n}\right)$$

$$= \left[\phi_{(X_i - \mu)}(t/n)\right]^n \quad \text{---} (1)$$

$$= \left[E(X_i - \mu)^0 + \frac{i(t/n)}{1!} E(X_i - \mu)^1 + \cdots\right]^n$$

$$= \left[1 + \frac{it}{n} E(X_i - \mu) + O(t/n) - \cdots\right]^n$$

$$\left[\because E(e^{itX}) = \mu_0' + \frac{it}{1!}\mu_1' + \cdots\right] \left[\because \mu_r = E(X - \mu)^r\right]$$

Appleying log on both sides

$\log \phi_{\bar{x}n}(t) = n \log\left(1 + \frac{it}{n} E(x_i - \mu) + o(t/n)\right]$

$\left[\because \log(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$

$= n\left[\frac{it}{n} E(x_i - \mu) + o(t/n) - \left[\frac{\frac{it}{n} E(x_i - \mu) + o\frac{t}{n}}{2}\right]^2 + \cdots\right.$

$= n\left[\frac{it}{n}[E(x_i - \mu)] + o(t/n)\right]$

$= it(\mu - \mu) + o(t/n)$

$= o(t/n) \to 0$

$\lim_{n\to\infty} \log \phi_{xn}(t) = \lim_{n\to\infty} o(t/n) = 0$

$\log \phi_{xn}(t) \to 0$ as $n \to \infty$

$\phi_{xn}(t) \to e^0 = 1$ as $n \to \infty$

$\phi_{xn}(t) \to \phi_x(t)$.

By using Levy's continuity theorem

$\Rightarrow F_{xn}(t) \to F_x(t)$

$\Rightarrow \therefore Y_n \xrightarrow{L} X : Y_n \xrightarrow{P} X = 0$ as $n \to \infty$

$\Rightarrow \frac{S_n - E(S_n)}{n} \xrightarrow{P} X = 0$

$\Rightarrow \frac{S_n}{n} \xrightarrow{P} E\left(\frac{S_n}{n}\right)$ which is WLLN.

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9) Properties of Riemann-Stieljes integral

1) If $f(x) \in R(\alpha)$ & $g(x) \in R(\alpha)$ on $[a,b]$ then $c_1 f + c_2 g \in R(\alpha)$ where c_1 & c_2 are constant ie $\int_a^b [c_1 f(x) + c_2 g(x)]\, d\alpha = x =$ $c_1 \int_a^b f(x)\, d\alpha(x) + c_2 \int_a^b g(x)\, d\alpha(x)$.

Proof - let $h = c_1 f + c_2 g$.

Given a partition P of $[a,b]$.

We can represent $S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\Delta\alpha_k$.

$$= \sum_{k=1}^{n} (c_1 f(t_k) + c_2 g(t_k))\Delta\alpha_k$$

$$= \sum_{k=1}^{n} c_1 f(t_k)\Delta\alpha_k + \sum_{k=1}^{n} c_2 g(t_k)\Delta\alpha_k$$

$$= c_1 \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\Delta\alpha_k$$

$$= c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha)$$

Since $f, g \in R(\alpha)$ on $[a, b]$.. Given $\varepsilon > 0$, choose $P_\varepsilon' \ni P_\varepsilon' \subseteq P$.

$$\Rightarrow \left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| < \frac{\varepsilon}{2|c_1|}$$

Simillarly choose $P_\varepsilon'' \ni P_\varepsilon'' \subseteq P$

$$\Rightarrow \left| S(P, g, \alpha) - \int_a^b g\, d\alpha \right| < \frac{\varepsilon}{2|c_2|}$$

let $P_\varepsilon = P_\varepsilon' \cup P_\varepsilon''$ then P is a finer than P_ε

We have $\left| S(P, h, \alpha) - \int_a^b f\, d\alpha - \int_a^b g\, d\alpha \right| \le |c_1| \cdot \frac{\varepsilon}{2|c_1|} + |c_2| \frac{\varepsilon}{2|c_2|}$

$$\left| S(P, h, \alpha) - \int_a^b f\, d\alpha - \int_a^b g\, d\alpha \right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon$$

$$\therefore \int_a^b (c_1 f + c_2 g)\, d\alpha \pm c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

2) If $f \in R(\alpha)$, $f \in R(\beta)$ on $[a, b]$ then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ for any 2 constant c_1 & c_2 we have $\int_a^b f d(c_1\alpha + c_2\beta = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta$.

Proof- let $P = \{x_0, x_1, \ldots x_n\}$ be the partition on a, b.

let $\gamma = c_1\alpha + c_2\beta$

$$\Delta\gamma_k = \gamma(x_k) - \gamma(x_{k-1})$$

$$= c_1\alpha(x_k) - c_2\beta(x_k) - c_1\alpha(x_{k-1}) - c_2\beta(x_{k-1})$$

$$= c_1(\alpha(x_k) - \alpha(x_{k-1})) + c_2(\beta(x_k)) - \beta(x_{k-1}))$$

$$= c_1\Delta\alpha_k + c_2\Delta\beta_k$$

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$$S(P,f,\gamma) = \sum_{k=1}^{n} f(x_k)\,\Delta\gamma_k$$

$$= \sum_{k=1}^{n} f(x_k)(c_1\Delta\alpha_k + c_2\Delta\beta_k)$$

$$= c_1\sum_{k=1}^{n} f(x_k)\Delta\alpha_k + c_2\sum_{k=1}^{n} f(x_k)\Delta\beta_k$$

$$= c_1 S(P,f,\alpha) + c_2 S(P,f,\beta)$$

Since $f \in R(\alpha)$, $f \in R(\beta)$ on $[a,b]$ for a given $\varepsilon>0$ $\exists$ a partition P_ε

$P_\varepsilon \ni P \subset P_\varepsilon$ & $\left|S(P,f,\alpha) - \int_a^b f\,d\alpha\right| \le \frac{\varepsilon}{2|c_1|}$

$$\left|S(P,f,\beta) - \int_a^b f\,d\alpha\right| \le \frac{\varepsilon}{2|c_2|}$$

Consider $\left|S(P,f,\gamma) - c_1\int_a^b f\,d\alpha - c_2\int_a^b f\,d\beta\right| \le c_1\left|S(P,f,\alpha) - \int_a^b f\,d\alpha\right| + c_2\left|S(P,f,\beta) - \int_a^b f\,d\beta\right|$

$$\le c_1\frac{\varepsilon}{2|c_1|} + c_2\frac{\varepsilon}{2|c_2|}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \Rightarrow \le \varepsilon$$

3) ST $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha$. Where $c \in [a,b]$

Proof - let us assume that $\int_a^c f\,d\alpha$ & $\int_a^b f\,d\alpha$ exists then we require to PT $\int_a^b f\,d\alpha$ also exists.

$$\int_a^c f\,d\alpha + \int_b^c f\,d\alpha = \int_a^b f\,d\alpha \quad \text{--- (1)}$$

let P be a partition on $[a,b] \ni c \in P$

let $P_1 = P \cap [a,c]$ & $P_2 = P \cap [c,b]$ then P_1 & P_2 are the Partition of a,c & c,b respectively.

$$S(P,f,\alpha) = S(P_1,f,\alpha) + S(P_2,f,\alpha)$$

$\Rightarrow \int_a^c f\,d\alpha$ exists for every $\varepsilon > 0$ $\exists$ a Partition P'_ε of $[a,c]$ $\ni$

$$|S(P_1, f, \alpha) - \int_a^c f\,d\alpha| \le \frac{\varepsilon}{2}$$

 $\int_c^b f\,d\alpha$ exists for every $\varepsilon > 0$ $\exists$ a Partition P''_ε of $[c,b]$ $\ni$

$$|S(P_2, f, \alpha) - \int_c^b f\,d\alpha| \le \frac{\varepsilon}{2}$$

let $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ then P_ε is a Partition on $[a,b]$

let $P = P_1 \cup P_2$ & $P'_\varepsilon \subseteq P_1$ & $P''_\varepsilon \subseteq P_2$ & P is a finer partition of P_ε.

Consider $|S(P, f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha|$

$$= |S(P_1, f, \alpha) + S(P_2, f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha|$$

$$\le |S(P_1, f, \alpha) - \int_a^c f\,d\alpha| + |S(P_2, f, \alpha) - \int_c^b f\,d\alpha|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \Rightarrow \le \varepsilon .$$

Applications of Probability theory

In probability theory the basic setup is a probability space (Ω, F, P) Here Ω is the outcome space ie the set of all possible outcomes for the random experiment we want to model. The set F is a collection of subsets of Ω, satisfying the 3 following Conditions.

1) $\phi \in F$ 2) $A \in F \Rightarrow A^c F$ 3) $A_i \in F$ for $i = 1, 2 \dots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

The function P is a probability measure defined on F ie a f^n $P: F \to [0,1] \ni$ 1) $P(\phi) = 0$ 2) $P(A^c) = 1 - P(A)$

3) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \phi$ when $i \ne j$

we suppose in ① that $A \in F$ & in ③ that $A_i \in F$ for every i for us to be able to apply P to the resulting sets, the corresponding conditions ② & ③ in the def^n of a σ algebra ensures that this is alright

A r.v X is a f^n $\Omega \to R \ni \{w : X(w) \le x\} \in F$. For every $x \in R$ this condition is Called measurability. $F(x) = P(w : X(w) \le x)$ Can be defined.