ng

① P/T $V(X) = E[V(X|Y)] + V[E(X|Y)]$ when $V(X|Y)$ and $E(X|Y)$ exists.

Proof :- Consider

$$E[V(X|Y)] + V[E(X|Y)]$$

($\because$ we know that $V(X) = E[X^2 - E(X)^2]$)

$$= E\left[E(X|Y)^2 - [E(X|Y)]^2\right] + E\left[[E(X|Y)]^2 - [E[E(X|Y)]]^2\right]$$

$$= E[E(X^2|Y)] - E[E(X|Y)]^2 + E[E(X|Y)]^2 - E[E(X)]^2$$

($\because E[E(X|Y)] = E(X)$)

$$= E(X^2) - [E(X)]^2$$

($\because E[E(X^2|Y)] = E(X^2)$)

$$= V(X)$$

③ S/T if $E(X^2) < \infty$ then $V(X) \geq V[E(X|Y)]$ equality holds iff X is a function of Y.

Proof :- From $V(X) = E[V(X|Y)] + V[E(X|Y)] \rightarrow ①$

WKT $V(X|Y) \geq 0$ then $E(V(X|Y)) \geq 0$

$\therefore$ from ① $\Rightarrow$ $V(X) \geq V(E(X|Y))$.

equality holds when $V(X) = V(E(X|Y))$

$$\therefore \quad E[V(X|Y)] = 0$$

$$E[E(X^2/Y) - [E(X|Y)]^2] = 0$$

$$E(X^2) = E[E(X|Y)]^2 = 0$$

$\therefore E(X^2) = E[E(X|Y)]^2$

$X^2 = [E(X|Y)]^2$

$X = E(X|Y)$

$\therefore$ X is a function of Y.

short

4 Define conditional expectations

Let X and Y be 2 r.v's on (Ω, B, P)

Let $h(x)$ be a real value borel measurable function.

further assume that $E[h(x)]$ exist then the conditional expectation of $h(x)$ given $X=y$ or $h(x) | Y=y$

ie $E(h(x) | Y=y) = \sum_x h(x)\ P(X=x | Y=y)$

$= \int_x h(x) \cdot f(x|y)\ dx$

Similarly $E(h(y) / X=x) = \sum_y h(y)\ P(Y=y | X=x)$

$= \int_y h(y) \cancel{f(x|y)} f(y|x)\ dy$

If $h(x) = x$ then

$E(x | Y=y) = \sum_x x \cdot P(X=x | Y=y)$

$= \int_x x \cdot f(x/y)\ dx$

Similarly $E(y | X=x) = \sum y \cdot P(Y=y | X=x)$

$= \int_y y\ f(y/x)\ dx$.

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that $E(E(X|Y)) = E(X)$; $E(X) < \infty$.

Proof:- Consider

$$E(E(X|Y)) = E\left[\sum_x x\, P(X=x \mid Y=y)\right]$$

$$= E\left[\sum_{i=1}^{m} x_i\, P(X=x_i \mid Y=y_j)\right]$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} x_i\, \frac{P[X=x_i \cap Y=y_j]}{\cancel{P(Y=y_j)}} \cdot \cancel{P(Y=y_j)}$$

$$= \sum\sum x_i\, P[X=x_i \cap Y=y_j]$$

$$= \sum_i x_i \sum_j P_{ij}$$

$$= \sum x_i\, P_{i.} \qquad \left(\because \sum_j P_{ij} = P_{i.}\right)$$

$$= E(X).$$

Q(i) If X_1 and X_2 are independent poisson random variable, with λ_1 and λ_2 then find $E[X_1 \mid X_1+X_2=n]$

Proof:- Consider $P(X_1=k \mid X_1+X_2=n)$

$$= \frac{P(X_1=k,\ X_1+X_2=n)}{P(X_1+X_2=n)}$$

$$= \frac{P(X_1=k,\ X_2=n-k)}{P(X_1+X_2=n)}$$

$$= \frac{\dfrac{e^{-\lambda_1}\lambda_1^{k}}{k!} \cdot \dfrac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= {}^nC_k \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} \quad ; k = 0, 1, \ldots n$$

which is binomial pmf with $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

Hence $E[X_1 \mid X_1+X_2] = n \cdot \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

($\because$ Mean of binomial $= np$)

(ii) S/T $E\left[\sum_{i=1}^{N} X_i\right] = E(n) \cdot E(X)$. $\quad$ ($= E(X)$)

proof :- Consider $E\left(\sum_{i=1}^{N} X_i\right) = E\left[E\left(\sum_{i=1}^{N} X_i / N\right)\right]$

$$= E\left[E\left(\sum_{i=1}^{n} X_i \mid N=n\right)\right]$$

$$= E\left[E\left(\sum_{i=1}^{n} X_i\right)\right]$$

$$= E\left[\sum_{i=1}^{n} E(X_i)\right]$$

$$= E[n \cdot E(X)]$$

(X_i and n_i are independent)

$$= E(n) \cdot E(X).$$

short

P/T $F(-\infty) = 0$ and $F(+\infty) = 1$

proof Consider

$$F(-\infty) = \lim_{x \to -\infty} F(x)$$

$$= \lim_{x \to -\infty} P(X \leq x)$$

$$= P\left[\lim_{x \to -\infty} (X \leq x)\right]$$

$$= P(X \leq -\infty).$$

$$= P(\phi) = 0$$

Consider $F(\infty) = \lim_{x \to \infty} F(x)$

$$= \lim_{x \to \infty} P(X \leq x)$$

$$= P(X \leq \infty)$$

$$= P(\Omega)$$

$$= 1$$

short

(5) Baye's theorem

Let $\{A_n\}$ be the sequence of disjoint events such that $P(A_n) > 0$, $\forall n = 1, 2 \ldots$ and $\sum_{n=1}^{\infty} A_n = \bigcup_{i=1}^{\infty} A_i = \Omega$. Let $B \in S$ with $P(B) > 0$ then $P(A_i|B) = \dfrac{P(A_i)\, P(B|A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B|A_i)}$, $\forall i = 1, 2 \ldots n$

proof:- Let $P(A_i|B) = \dfrac{P(A_i \cap B)}{P(B)} \rightarrow (*)$ (from conditional probability)

$P(A_i \cap B) = P(A_i) \cdot P(B|A_i) \rightarrow (1)$ (from multiplicative law)

$$B = \bigcup_{i=1}^{\infty} (A_i \cap B) \quad \text{(from distributive law)}$$

$$B = \sum_{i=1}^{\infty} (A_i \cap B)$$

$$P(B) = \sum_{i=1}^{\infty} P(A_i \cap B)$$

$$P(B) = \sum_{i=1}^{\infty} P(A_i) \cdot P(B|A_i) \longrightarrow (2)$$

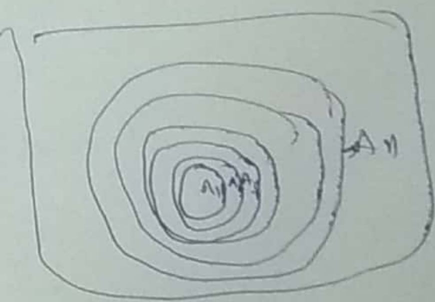
Sub. (1) and (2) in eq (*)

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{i=1}^{\infty} P(A_i) \cdot P(B|A_i)}.$$

(5) Let $\{A_n\}$ be a seq^n of events defined on S. then

(a) If $\{A_n\}$ is a non-decreasing seq^n. of events

S/T $\quad \lim_{n \to \infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$.



Proof :- Consider $A = \bigcup_{j=1}^{\infty} A_j$

from the figure WKT

$$A_2 = A_1 \cup (A_2 - A_1)$$

$$A_3 = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2)$$

$$A = A_n \cup \left[\bigcup_{i=n}^{\infty} (A_{j+1} - A_j)\right]$$

$$P(A) = P(A_n) + P\left[\bigcup_{i=1}^{\infty} (A_{j+1} - A_j)\right]$$

$$P(A) = P(A_n) + \sum_{j=n}^{\infty} P(A_{j+1} - A_j)$$

$\rightarrow$ as $n \to \infty$

$$P(A) = \lim_{n\to\infty} P(A_n) + \underbrace{\lim_{n\to\infty} \sum_{j=n}^{\infty} P(A_{j+1} - A_j)}_{0}$$

$$P(A) = \lim_{n\to\infty} P(A_n) \longrightarrow (1)$$

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A_n is a finite Non-decreasing seqn of real no's such that $A_{n-1} \leq A_n$ then

$$A = \bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{\infty} A_j$$

where 'A' is known as the limit of the seq. A_n

$$A = \lim_{n\to\infty} A_n$$

$$P(A) = P\left(\lim_{n\to\infty} A_n\right) \rightarrow (2)$$

WKT $A = \bigcup_{j=1}^{\infty} A_j$

$$P(A) = P\left(\bigcup_{j=1}^{\infty} A_j\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right) \rightarrow (3)$$

from (1), (2), (3)

$$\cancel{P(A) = P\left(\lim_{n\to\infty} P(A_n)\right) = \left\{\lim_{n\to\infty} P(A_n)\right\}}$$

$$P(A) = P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(b) ~~S/T~~ If $\{A_n\}$ be a non-increasing seq of events then S/T. $\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$.

Proof: From above theorem

$$\lim_{n\to\infty} P(A_n^c) = P\left(\lim_{n\to\infty} A_n^c\right) = P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$

$\Rightarrow$ consider $\lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty} (1 - P(A_n))$

$$= 1 - \lim_{n\to\infty} P(A_n) \rightarrow (1)$$

Consider $P\left(\underset{n\to\infty}{Lt} A_n^c\right) = P \underset{n\to\infty}{Lt}(\Omega - A_n)$

$= P(\Omega) - P\left(\underset{n\to\infty}{Lt} A_n\right)$

$= 1 - P\left(\underset{n\to\infty}{Lt} A_n\right) \longrightarrow (2)$

Consider $P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = P\left[\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right]$

$= 1 - P\left(\bigcap_{n=1}^{\infty} A_n\right) \longrightarrow (3)$

from (1), (2), (3) we have

$1 - \underset{n\to\infty}{Lt} P(A_n) = 1 - P\left(\underset{n\to\infty}{Lt} A_n\right) = 1 - P\left(\bigcap_{n=1}^{\infty} A_n\right)$

$\therefore \underset{n\to\infty}{Lt} P(A_n) = P\left(\underset{n\to\infty}{Lt} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right).$

(6) Show If $f \in R(\alpha)$, $g \in R(\alpha)$ on $[a,b]$

S/T $\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g \cdot d\alpha.$

proof:- Let $h = c_1 f + c_2 g$

Let 'P' be the partition on $[a,b]$

then $S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\, \Delta\alpha_k$

$= \sum_{k=1}^{n} \left(c_1 f(t_k) + c_2 g(t_k)\right) \Delta\alpha_k$

$= c_1 \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\, \Delta\alpha_k$

$= c_1\, S(P, f, \alpha) + c_2\, S(P, g, \alpha)$

since $f \in R(\alpha)$, $g \in R(\alpha)$ on $[a,b]$

(5)

...se $P'_\epsilon \ni P'_\epsilon \subseteq P$. $\because \epsilon > 0$

$$\Rightarrow \left| S(P, f, \alpha) - \int_a^b f\, d\alpha \right| \leq \frac{\epsilon}{2|C_1|} \longrightarrow (1)$$

similarly choose $P''_\epsilon \ni P''_\epsilon \subseteq P$

$$\left| S(P, g; \alpha) - \int_a^b g\, d\alpha \right| \leq \frac{\epsilon}{2|C_2|} \longrightarrow (2)$$

~~Let~~

$\Rightarrow C_1 (1) + C_2 (2)$

$$\left| C_1 S(P, f, \alpha) - C_1 \int_a^b f\, d\alpha + C_2 S(P, g, \alpha) - C_2 \int_a^b g\, d\alpha \right| \leq \frac{\epsilon C_1}{2 C_1} + \frac{\epsilon C_2}{2 C_2}$$

$$\left| S(P, h, \alpha) - C_1 \int_a^b f\, d\alpha - C_2 \int_a^b g\, d\alpha \right| \leq \epsilon$$

$$\& \; S(P, h, \alpha) = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b g\, d\alpha$$

$$\Sigma h(t_k) \Delta\alpha_k = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b g\, d\alpha$$

$$\int_a^b h\, d\alpha = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b g\, d\alpha .$$

$$\int_a^b (C_1 f + C_2 g)\, d\alpha = C_1 \int_a^b f\, d\alpha + C_2 \int_a^b g\, d\alpha$$

(7) compute $E[Y | X = x]$ and $V(Y | X = x)$ for bivariate dist with

$$f(x, y) = \begin{cases} 8xy & ; 0 \leq x < y < 1 \\ 0 & ; \text{o.w} \end{cases}$$

(sol) $E[Y | X = x] = \int_y y \cdot f(y|x)\, dy$

consider $f(y|x) = \dfrac{f(x,y)}{f(x)}$ ~~$= \dfrac{8xy}{}$~~

consider $f(x) = \int_y f(x,y)\, dy = \int_x^1 8xy\, dy = \overset{4}{\cancel{8}} x \left[\frac{y^2}{\cancel{2}} \right]_x^1 =$

$= 4x(1 - x^2).$

$$\therefore f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{8}^{2}xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

$$\therefore E(y/x=x) = \int_y y \cdot f(y/x)\,dy$$

$$= \int_x^1 y \cdot \frac{2y}{1-x^2}\,dy$$

$$= \frac{2}{1-x^2}\int_x^1 y^2\,dy = \frac{2}{1-x^2}\left[\frac{y^3}{3}\right]_x^1 = \frac{2}{1-x^2}\left[\frac{1-x^3}{3}\right]$$

$$= \frac{2(1-x)(1+x^2+x)}{3(1+x)(1-x)}$$

$$= \frac{2(1+x^2+x)}{3(1+x)}.$$

$$\therefore V(y/x=x) = E(y^2/x=x) - [E(y/x=x)]^2$$

$$= \int_y y^2 f(y/x)\,dy - \left[\frac{2}{3}\left(\frac{x^2+x+1}{1+x}\right)\right]^2$$

$$= \int_y y^2\,\frac{2y}{1-x^2}\,dy - \frac{4}{9}\,\frac{(x^2+x+1)^2}{(1+x)^2}$$

$$= \frac{2}{1-x^2}\left[\frac{y^4}{4}\right]_x^1 - \frac{4}{9}\,\frac{(x^2+x+1)^2}{(1+x)^2}$$

$$= \frac{\cancel{2}}{1-x^2}\left(\frac{1-x^4}{\cancel{4}_2}\right) - \frac{4}{9}\,\frac{(x^4+x^2+1+2x^3+2x+2x^2)}{(1+x)^2}$$

$$= \frac{1+x^2}{2} - \frac{4}{9}\,\frac{(x^4+3x^2+2x^3+2x+1)}{(1+x)^2}$$

$$= \frac{9(1+x^2+2x)(1+x^2) - 8(x^4+2x^3+3x^2+2x+1)}{18(1+x)^2}$$

$$= \frac{9(x^4+2x^3+2x^2+2x+1) - 8(x^4+2x^3+3x^2+2x+1)}{18(1+x)^2}$$

$$= \frac{x^4+2x^3-6x^2+2x+1}{18(1+x)^2}$$

4