PT - 4 ①

Laws of large numbers

Let $\{X_n ; n \geq 1\}$ be a sequence of large number defined on (Ω, H, P)

Let S_n be $S_n = \sum_{n=1}^{\infty} X_n$

Let $\{A_n ; n \geq 1\}$ and $\{B_n ; n \geq 1\}$ be 2 sequence of real number's such that $B_n \geq 0, \forall n \geq 1$ and $B_n \to \infty$ as $n \to \infty$ then $\{X_n\}$ is said to follow

(i) Weak law of large no's (WLLN)

If $\frac{S_n - A_n}{B_n} \to 0$ as $n \to \infty$

(ii) strong law of large no's (SLLN)

If $\frac{S_n - A_n}{B_n} \xrightarrow{a.s.} 0$ as $n \to \infty$

(iii) central limit theorem (CLT)

If $\frac{S_n - A_n}{B_n} \xrightarrow{L} Z \sim N(0,1)$ as $n \to \infty$

where A_n are known as centering constant

B_n ~~are~~ known as scaling constant.

WLLNS

① **Bernoulli's WLLNS**

St:- Let $\{X_n ; n \geq 1\}$ be a seq. of iid bernoulli random variables defined as $P(X_n = 1) = P$, $P(X_n = 0) = 1 - P$
$= q = 0$
$\forall n \geq 1$ then the seq. $\{X_n\}$ satisfies WLLN.

$$\Rightarrow \quad \frac{S_n}{n} \xrightarrow{P} E\left[\frac{S_n}{n}\right] \text{ as } n \to \infty$$

$$\frac{S_n - A_n}{B_n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty$$

Proof:- Let $S_n = \sum_{i=1}^{n} X_i$; $A_n = E(S_n)$; $B_n = n$

WKT $X_n \sim B(1, P)$

$$E(X_n) = 0 \times (1-P) + 1 \times P$$
$$= P, \quad \forall n \geq 1$$

X_n	0	1
$P(X_n)$	$1-P$	P

$$E(X_n^2) = 0^2 \times (1-P) + 1^2 \times P$$
$$= P,$$

$$V(X_n) = E(X_n^2) - [E(X_n)]^2$$
$$= P - P^2 = P(1-P) = Pq$$

consider $E\left(\frac{S_n}{n}\right) = \frac{1}{n} E(S_n) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$

$$= \frac{1}{n} \sum_{i=1}^{n} P = \frac{1}{n} \cdot nP = P$$

$$V\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(S_n) = \frac{1}{n^2} \cdot n \cdot Pq = \frac{Pq}{n}.$$

Consider

$$P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| > \epsilon\right] \leq \frac{V\left(\frac{S_n}{n}\right)}{\epsilon^2}$$

$$P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| > \epsilon\right] \leq \frac{pq/n}{\epsilon^2}$$

$$\leq \frac{pq}{n\epsilon^2}$$

$$\leq 0 \text{ as } n \to \infty$$

$$P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| > \epsilon\right] \to 0 \text{ as } n \to \infty$$

$$\frac{S_n}{n} \xrightarrow{P} E\left(\frac{S_n}{n}\right) \text{ as } n \to \infty$$

(or)

$$P\left[\left|\frac{S_n}{B_n} - \frac{A_n}{B_n}\right| > \epsilon\right] \to 0, \text{ as } n \to \infty$$

$$\frac{S_n - A_n}{B_n} \xrightarrow{P} 0 \text{ as } n \to \infty$$

$\therefore$ The seq. of Bernoulli r.v's $\{X_n\} \sim$ WLLN.

② Khintchine's WLLNS

St:- Let $\{X_n ; n \geq 1\}$ be a seq. of iid r.v's Let $Y_n = \frac{S_n - E(S_n)}{n}$ with characteristic function $\phi_{Y_n}(t)$.

and with common mean $\mu < \infty$ then

$$Y_n \xrightarrow{P} X \text{ as } n \to \infty$$

Proof: Given

$$x_n = \frac{S_n - E(S_n)}{n} = \frac{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu}{n} = \cancel{\frac{\sum x_i - n\mu}{n}}$$

$$= \cancel{\frac{\sum_{i=1}^{n} (x_i - \mu)}{n}}$$

$$= \frac{\sum_{i=1}^{n} x_i - E\left(\sum_{i=1}^{n} x_i\right)}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} [E(x_i)]}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i - n\mu}{n}$$

$$= \frac{\sum_{i=1}^{n} (x_i - \mu)}{n}$$

consider $\phi_{x_n}(t) = E\left(e^{it x_n}\right)$

$$= E\left[e^{it\left(\frac{\sum_{i=1}^{n}(x_i-\mu)}{n}\right)}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{i\left(\frac{t}{n}\right)\cdot(x_i-\mu)}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{i\left(\frac{t}{n}\right)(x_i-\mu)}\right]$$

$$= \prod_{i=1}^{n} \phi_{(x_i-\mu)}\left(\frac{t}{n}\right).$$

$$= \left[\phi_{(x_i-\mu)}(t/n)\right]^n \longrightarrow (1)$$

$$= \left[E(x_i-\mu)^0 + \frac{i(t/n)}{1!} E(x_i-\mu)^1 + \cdots + \cdots\right]^n$$

$$\phi_{x_n}(t) = \left\{1 + \frac{it}{n} E(x_i-\mu) + O(t/n) \cdots\right]^n$$

$$\left(\because E(e^{itx}) = \mu_0' + \frac{it}{1!}\mu_1' + \cdots \quad \therefore \mu_r = E(x-\mu)^r \cdots\right)$$

Applying log on B.S

③

$$\log \phi_{Y_n}(t) = n\log\left(1 + \frac{it}{n} E(X_i - \mu) + o(t/n)\right)$$

$$\left(\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$= n\left[\frac{it}{n} E(X_i - \mu) + o\cdot(t/n) - \frac{\left[\frac{it}{n} E(X_i - \mu) + o\cdot \frac{t}{n}\right]^2}{2} + \cdots\right]$$

$$= n\cdot\left[\frac{it}{n}\left[E(X_i) - \mu\right] + o\cdot(t/n).\right]$$

$$= it\,(\mu - \mu) + o\cdot(t/n)$$

$$= o\cdot(t/n).$$

$$\lim_{n\to\infty} \log \phi_{Y_n}(t) = \lim_{n\to\infty} o(t/n) = 0$$

$$\log \phi_{Y_n}(t) \longrightarrow 0 \quad \text{as } n\to\infty$$

$$\phi_{Y_n}(t) \longrightarrow e^0 = 1 \quad \text{as } n\to\infty$$

$$\phi_{Y_n}(t) \longrightarrow \phi_X(t).$$

By using levy's continuity theorem.

$$\Rightarrow F_{Y_n}(t) \longrightarrow F_X(t)$$

$$\Rightarrow Y_n \xrightarrow{L} X\ ;\quad Y_n \xrightarrow{P} X = 0\ . \text{ as } n\to\infty$$

$$\Rightarrow \frac{S_n - E(S_n)}{n} \xrightarrow{P} X = 0$$

$$\Rightarrow \frac{S_n}{n} \xrightarrow{P} E\left(\frac{S_n}{n}\right) \text{ which is WLLN.}$$

3 kolmogorov inequality

st:- If the $\{X_n ; n \geq 1\}$ is the seq. of square integrable. independent random variable

i.e $E(X_n^2) = \sigma_n^2 < \infty$, $\forall n \geq 1$ then

$$P\left[\max_{1 \leq k \leq n} |S_k - E(S_k)| > \epsilon\right] \leq \frac{1}{\epsilon^2} \sum_{k=1}^{n} \sigma_k^2 ; \quad \epsilon > 0$$

where σ_k^2 is variance of X_k ; $k = 1, 2 \ldots n$ and

$$S_k = X_1 + X_2 + \cdots + X_k .$$

proof:- If $k=1$ Then $|S_1 - E(S_1)| = |X_1 - E(X_1)|$

$k=2$ Then $|S_2 - E(S_2)| = |X_1 + X_2 - E(X_1 + X_2)|$

In general if $k=k$ Then

$$|S_k - E(S_k)| = |(X_1 + X_2 + \cdots + X_k) - E(X_1 + X_2 + \cdots + X_k)|$$

$$= \left| \sum_{i=1}^{k} X_i - E\left(\sum_{i=1}^{k} X_i\right) \right|$$

$$[|S_n - E(S_n)| > \epsilon] \geq \left[\max_{1 \leq k \leq n} (|S_k - E(S_k)| > \epsilon)\right] \to (1)$$

By using chebychev's inequality, we have

$$P[|S_n - E(S_n)| > \epsilon] \geq P\left[\max_{1 \leq k \leq n} |S_k - E(S_k)| > \epsilon\right]$$

$$\leq \frac{V(S_n)}{\epsilon^2}$$

$$\leq \frac{\sum_{i=1}^{n} V(X_k)}{\epsilon^2} \quad \cancel{\frac{\sum_{i=1}^{n} \sigma_k^2}{\epsilon^2}}$$

$$\leq \frac{\sum_{i=1}^{n} \sigma_k^2}{\epsilon^2} .$$

$\therefore$ ~~P max~~

$$\therefore P\left[\max_{1 \leq k \leq n} |S_k - E(S_k)| > \epsilon\right] \leq \frac{1}{\epsilon^2} \sum_{i=1}^{n} \sigma_k^2$$

(4) Kolmogorov's SLLNs for independent Random variable (4)

St: Let the sequence $\{X_n ; n \geq 1\}$ be a sequence of square integrable independent r.v and if $\frac{1}{n^2}\sum_{n=1}^{\infty} \sigma_n^2 < \infty$ Then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i)) \xrightarrow{a.s} 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \frac{S_n}{n} - E\left(\frac{S_n}{n}\right) \xrightarrow{as} 0 \quad \text{as } n \to \infty$$

$$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right) \quad \text{as } n \to \infty$$

$\{x_n\}$. obey SLLN

Proof: Given $\{X_n ; n \geq 1\}$ is a seq. of iid r.v's and

$$\frac{1}{n^2}\sum_{i=1}^{n} \sigma_n^2 < \infty, \ \forall n \geq 1 \longrightarrow (1)$$

from Lemma (1) (if $\sum_{n=1}^{\infty} b_n < \infty$ then, $\frac{1}{n}\sum_{k=1}^{n} k.b_k \to 0$ as $n \to \infty$)

$\sum_{n=1}^{\infty}(X_n - E(X_n))$ converges almost surely to a real valued r.v where $\sum_{n=1}^{\infty}\sigma_n^2 < \infty$

$\because$ Convergence of infinite series $\Leftrightarrow$ Convergence of partial series.

$\Rightarrow \sum_{i=1}^{n}(X_i - E(X_i))$ converges almost surely to a real valued r.v when $\sum_{i=1}^{n}\sigma_i^2 < \infty$

$\sum_{i=1}^{n}\left(\frac{X_i - E(X_i)}{i}\right)$ converges almost surely (a.s), $\forall n \geq 1$

from lemma (1) If $\sum_{n=1}^{\infty} b_n < \infty$ then

$$\frac{1}{n}\sum_{k=1}^{n} k b_k \longrightarrow 0 \quad \text{as } n \to \infty$$

(consider $k = i$

($\because b_i = \frac{X_i - E(X_i)}{i}$))

$$\frac{1}{n}\sum_{i=1}^{n} i\, b_i$$

$$\frac{1}{n}\sum_{i=1}^{n} i\left(\frac{X_i - E(X_i)}{i}\right) \xrightarrow{a.s} 0 \quad \text{as } n \to \infty$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i)) \xrightarrow{a.s} 0, \quad \text{as } n \to \infty$$

$$\frac{S_n}{n} - E\left(\frac{S_n}{n}\right) \xrightarrow{a.s} 0, \quad \text{as } n \to \infty$$

$$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right) \text{ as } n \to \infty$$

where

$A_n = E(S_n)$, $B_n = n$

$\{X_n\}$ satisfies SLLN.

⑤ Kolmogorov's SLLNS for iid random variables

If $\{X_n ; n \geq 1\}$ is a seq. of iid integrable r.v's with common mean $\mu < \infty$ then the $\{X_n\}$ obeys SLLN.

$$\Rightarrow \frac{S_n}{n} \xrightarrow{a.s} \mu \text{ as } n \to \infty$$

Proof:- Given $\{X_n ; n \geq 1\}$ is a seq. of iid r.v's with common mean $\mu < \infty$

consider $S_n = \sum_{i=1}^{n} X_i$ $\forall n \geq 1$, $E(S_n) = \sum x_i = n\mu$.

This S_n is
If $\{X_n\}$ is an integrable iid seq. Then.

$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right)$ as $n \to \infty$ According to Kolmogorov's SLLN

$$\frac{S_n}{n} \xrightarrow{a.s} \frac{n\mu}{n} \text{ as } n \to \infty$$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu \text{ as } n \to \infty.$$

(6) Borel's SLLN / Bernoulli's / special case of Kolmogorov's SLLN (5)

St:- If $\{X_n, n \geq 1\}$ is the seq. of iid Bernoulli's r.v

then. $P(X_n = 1) = p$, $P(X_n = 0) = 1 - p = q$, $\forall n \geq 1$

then $\frac{S_n}{n} \to E\left(\frac{S_n}{n}\right)$ as $n \to \infty$

$\{X_n\}$ obeys SLLN

Proof :- Given $X_n \sim B(1, p)$, $\forall n \geq 1$ then

$$E(X_n) = p, \quad V(X_n) = pq < \infty, \quad \forall n \geq 1$$

The seq. r.v is a square integrable

Consider $\frac{1}{n^2} \sum_{n=1}^{\infty} \sigma_n^2 = \frac{1}{n^2} \sum_{n=1}^{\infty} V(X_n)$

$$= \frac{1}{n^2} \cdot \sum_{n=1}^{\infty} pq$$

$$= pq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

from p-test infinite series any series $\sum \frac{1}{n^p}$ is converges if $p > 1$ and divergence if $p < 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent $< \infty$

$$\therefore pq \sum_{i=1}^{\infty} \frac{1}{n^2} < \infty$$

from Kolmogorov's SLLN

The $\{X_n; n \geq 1\}$ obey SLLN

$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right)$ as $n \to \infty$

$\frac{S_n}{n} \xrightarrow{a.s} \frac{np}{n}$ as $n \to \infty$

$\frac{S_n}{n} \xrightarrow{a.s} p$ as $n \to \infty$

$E\left(\frac{S_n}{n}\right) = \frac{1}{n} E\left(\sum X_i\right)$

$= \frac{1}{n} \sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n} \cdot n \cdot p$)

(7) Demoviere-Laplace form of CLT

Let $\{X_n, n \geq 1\}$ be the seq. of independent r.v with the $P(X_k = 1) = P$, $P(X_k = 0) = 1-P$ Then the cummulative sum of the square.

i.e $S_n = \sum_{k=1}^{n} X_k$ is asymptotically Normal $\frac{S_n - E(S_n)}{SD(S_n)} \xrightarrow{L} N(0,1)$ as $n \to \infty$

Proof :- Let X_n be the seq. follows Bernoullis Distribution. Let X_n be the seq. follows Bernoulli's distribution then it mean and variance are with the parameter 'p'

$$E(X_K) = P \text{ and } V(X_K) = Pq$$

Then it's mgf will be $M_{X_K}(t) = (q + pe^t)$

Let $S_n = \sum_{k=1}^{n} X_k$ be the cummulative sum of Bernoullis r.v which follows Binomial dist. with the parameter n p and n Then the mgf will be

$$M_{S_n}(t) = (q + pe^t)^n$$

its mean and variance are $E(S_n) = np$ and $V(S_n) = npq$

The standard sum of the seq. is denoted by

$$Z = \frac{S_n - E(S_n)}{\sqrt{V(S_n)}} = \frac{S_n - np}{\sqrt{npq}}$$

Then the mgf of Z is

$$M_Z(t) = E\left[e^{t\left(\frac{S_n - np}{\sqrt{npq}}\right)}\right]$$

$$= e^{\frac{-np}{\sqrt{npq}} t} \cdot E\left(e^{t\left(\frac{S_n}{\sqrt{npq}}\right)}\right)$$

$$= e^{\frac{-np}{\sqrt{npq}} t} \cdot M_{S_n}\left(t/\sqrt{npq}\right)$$

$$= e^{-\frac{np}{\sqrt{npq}} t} \left[q + pe^{\frac{t}{\sqrt{npq}}}\right]^n$$

$$= \left[e^{-\frac{p}{\sqrt{npq}} t} \cdot \left(q + pe^{\frac{t}{\sqrt{npq}}}\right)\right]^n$$

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$$M_Z(t) = \left[q e^{\frac{-p}{\sqrt{npq}} t} + p e^{\frac{t-pt}{\sqrt{npq}}}\right]^n$$

$$= \left[q \cdot e^{\frac{-p}{\sqrt{npq}} t} + p e^{\frac{q}{\sqrt{npq}} t}\right]^n$$

$$= \left[q\left\{1 - \frac{pt}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{pt}{\sqrt{npq}}\right)^2 + 0(n^{-3/2})\right\} + p\left\{1 + \frac{qt}{\sqrt{npq}} + \frac{1}{2!}\left(\frac{qt}{\sqrt{npq}}\right)^2 + 0\cdot(n^{-3/2})\right]^n$$

$$= \left[1 + \frac{t^2}{2n} + 0(n^{-3/2})\right]^n$$

$\frac{p^2t^2}{npq}$

$\frac{pt^2}{2n}$

$$\underset{n\to\infty}{Lt} M_Z(t) = \underset{n\to\infty}{Lt}\left[1 + \frac{t^2}{2n} + 0\cdot(n^{-3/2})\right]^n \simeq e^{t^2/n}$$

$\therefore$ It is the mgf of standard normal variate

$\therefore$ for the uniqueness theorem of mgf $Z \sim N(0,1)$

$$Z = \frac{S_n - E(S_n)}{S.D(S_n)} \longrightarrow N(0,1) \text{ as } n \to \infty$$

8 Levy-lindeberg form of CLT

If the $\{x_n, n \geq 1\}$ is a seq. of iid r.v with the $E(x_i) = \mu$, $V(x_i) = \sigma^2 < \infty$, $i = 1, 2 \ldots$ Thus the $\{x_n\}$ obeys central limit theorem.

ie $$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \xrightarrow{L} Z \sim N(0,1) \text{ as } n \to \infty$$

where $$\frac{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu}{\sigma\sqrt{n}} \xrightarrow{L} Z \sim N(0,1) \text{ as } n \to \infty$$

proof :- Consider $\phi_n(t)$ as a characteristic function of

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} = \frac{\sum_{i=1}^{n}(x_i - \mu)}{\sigma\sqrt{n}}$$

$$\phi_n(t) = E\left[e^{it\left(\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma\sqrt{n}}\right)}\right]$$

$$= E\left[e^{\frac{it}{\sigma\sqrt{n}}\sum_{i=1}^{n}(x_i-\mu)}\right]$$

Let $\frac{t}{\sigma\sqrt{n}} = t'$

$$= E\left[e^{it'\sum_{i=1}^{n}(x_i-\mu)}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{it'(x_i-\mu)}\right]$$

$$= \prod_{i=1}^{n} \phi_{(x_i-\mu)}(t')$$

$$= \left(\phi_{(x_i-\mu)}(t')\right]^n$$

$$= \left[1 + it'E(x_i-\mu) + \frac{(it')^2}{2!}E(x_i-\mu)^2 + O\left(\frac{it'}{n}\right)\right]^n$$

$$= \left[1 + 0 - \frac{(t')^2}{2}E(x_i-\mu)^2 + O\left(\frac{it'}{n}\right)\right]^n$$

Applying log on B.S

$$\log\phi_n(t) = \log\left[1 - \frac{(t')^2}{2}E(x_i-\mu)^2 + O\left(\frac{it'}{n}\right)\right]^n$$

$$= n\log\left[1 - \left(\frac{(t')^2}{2}\sigma^2 + O\left(\frac{it'}{n}\right)\right)\right]$$

$$= n\left[-\left(\frac{t'^2\sigma^2}{2} + O(t')^2\right) - \frac{\left(\frac{t'^2\sigma^2}{2} + O(t')^2\right)^2}{2} \ldots\right]$$

$$= -n\left[\frac{t'^2\sigma^2}{2} + O(t')^2\right] = \cancel{-n\left[\frac{(t/\sigma\sqrt{n})^2\sigma^2}{2} + O\left(\frac{t}{\sigma\sqrt{n}}\right)\right]}$$

$\cancel{\lim_{n\to\infty}\log\phi_n(t) = -n}$

$$= -n\left[\frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2\sigma^2}{2} + O\left(\frac{t}{\sigma\sqrt{n}}\right)^2\right]$$

$$= -\frac{t^2}{2} + O\left(\frac{t}{\sigma^2}\right).$$

$$\lim_{n\to\infty} \log \phi_n(t) = \lim_{n\to\infty}\left(-\frac{t^2}{2} + o\left(\frac{t}{\sigma^2}\right)\right)$$

$$\log \phi_n(t) = -\frac{t^2}{2}$$

$$\therefore \phi_n(t) = e^{-t^2/2}$$

where $e^{-t^2/2}$ is $\phi_z(t)$ where z is standard normal variate

$$\Rightarrow \phi_n(t) = \phi_z(t).$$

By levy continunity Theorem

$$F_n(x) = F_z(t) \quad \text{as } n\to\infty$$

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \xrightarrow{L} z \quad \text{as } n\to\infty$$

$\therefore$ The $\{X_n\}$ satisfy the central limit Theorem

9 Liapunov's form of CLT

Let $\{X_n ; n\geq 1\}$ be a seq. of independent rv's then there exist a +ve number $\delta \ni$ as $n\to\infty$;

$$\frac{1}{B_n^{2+\delta}} \sum_{k=1}^{n} E\left[|X_k - \mu_k|^{2+\delta}\right] = \frac{V_{2+\delta}}{(B_n)^{2+\delta}} \to 0 \quad \to (1)$$

then $\lim_{n\to\infty} P\left(\frac{S_n - E(S_n)}{B_n} \leq x\right) \xrightarrow{L} \phi(x) \sim N(0,1)$

Proof:- condition (1) is similar to Lindberg feller condition

Hence ~~condis~~ consider Lindberg feller condition.

$$g_n(\epsilon) = \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{A_n} |x_k - \mu_k|^2 f_k(x)\, dx_k$$

$$A_n = [|x_k - \mu_k| > \epsilon B_n]$$

$$= \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{A_n} \frac{|x_k - \mu_k|^{2+\delta}}{|x_k - \mu_k|^{\delta}} f_k(x)\, dx_k$$

$$< \frac{B_n^{-2}}{(\epsilon B_n)^{\delta}} \sum_{k=1}^{n} \int_{A_n} |x_k - \mu_k|^{2+\delta} f_k(x)\, dx_k$$

$$\because \frac{1}{|x_k - \mu_k|} < \frac{1}{\epsilon B_n}$$

$$\leq \frac{1}{\epsilon^{\delta} B_n^{2+\delta}} \sum_{k=1}^{n} \int_{-\infty}^{\infty} (x_k - \mu_k|^{2+\delta} f_k(x)\, dx_k$$

$$\left(\because \int_A f(x)\,dx \leq \int_{-\infty}^{\infty} f(x)\,dx\right)$$

$$\leq \frac{1}{\epsilon^{\delta} B_n^{2+\delta}} \sum_{k=1}^{n} E\left[|x_k - \mu_k|^{2+\delta}\right] \to 0$$

$$g_n(\epsilon) \leq \frac{1}{B_n^{2+\delta}} \sum_{k=1}^{n} E\left[|x_k - \mu_k|^{2+\delta}\right] \to 0 \text{ as } n \to \infty$$

$$= \frac{V_{2+\delta}}{B_n^{2+\delta}} \to 0 \text{ as } n \to \infty$$

$\{x_n\}$ obeys CLT

$\Rightarrow g_n(\epsilon) \to 0$ as $n \to \infty$

$\therefore$ $\{x_n\}$ obeys CLT.

$\frac{V_{2+\delta}}{B_n^{2+\delta}} \to 0$ as $n \to \infty$

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(10) Lindberg Feller form of CLT

St:- let $\{X_n, n \geq 1\}$ be the seq of independent degenerate r.v with the distinct function. $F_1, F_2, \ldots F_n$. such that which exists $\forall n \geq 1$ and

$E(X_n) = \mu_n$, $V(X_n) = \sigma_n^2 < \infty$

for any $\epsilon > 0$

$$g_n(\epsilon) = \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{|x_k - \mu_k| > \epsilon B_n} (x_k - \mu_k)^2 \, dF_k(x) \longrightarrow 0$$

where $B_n^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$, $\forall n \geq 1$ Then

(a) Then CLT holds for $\{x_n\} \Rightarrow \frac{S_n - E(S_n)}{B_n} \xrightarrow{L} Z$

$$\left(\frac{S_n - E(S_n)}{B_n}\right) \longrightarrow F_Z(x) \text{ as } n \to \infty$$

(b) ~~Max~~ $\max_{1 \leq k \leq n} \frac{\sigma_k^2}{B_n^2} \longrightarrow 0$ as $n \to \infty$.