(i) the r the point goes out of the control limits at the xth sample and the probability of this is p_n, and

(ii) In the remaining $(x-1)$ samples, exactly $(r-1)$ points go out of the control limits, and its probability is :

$$ {}^{x-1}C_{r-1}\, p_n^{\,r-1}(1-p_n)^{x-r} $$

Hence, by the compound probability theorem, the required probability is given by

$$ P(E) = P(i).\, P(ii) = p_n \cdot {}^{x-1}C_{r-1}\, p_n^{\,r-1}(1-p_n)^{x-r} $$

$$ = \left(p_n/1-p_n\right)^{r}\, {}^{x-1}C_{r-1}(1-p_n)^{x}\,;\ x \ge r $$

1·9. CONTROL CHART FOR ATTRIBUTES

In spite of wide applications of $\overline{X}$ and R-(or σ) charts as a powerful tool of diagnosis of sources of trouble in a production process, their use is restricted because of the following limitations :

1. They are charts for variables only, *i.e.*, for quality characteristics which can be measured and expressed in numbers.

2. In certain situations they are impracticable and un-economical, *e.g.*, if the number of measurable characteristics, each of which could be a possible candidate for $\overline{X}$ and R charts, is too large, say 30,000 or so then obviously there can't be 30,000 control charts.

As an alternative to $\overline{X}$ and R-charts, we have the control chart for attributes which can be used for quality characteristics :

(i) which can be observed only as attributes by classifying an item as defective or non-defective *i.e.*, conforming to specifications or not, and

(ii) which are actually observed as attributes even though they could be measured as variables, *e.g.*, go and no-go gauge test results.

There are two control charts for attributes :

(a) Control chart for fraction defective (p-chart) or the number of defectives (np or d chart).

(b) Control chart for the number of defects per unit (c-chart).

1·9·1. Control Chart for Fraction Defective (p-chart). While dealing with attributes, a process will be adjudged in statistical control if all the samples or sub-groups are ascertained to have the same population proportion P.

If 'd' is the number of defectives in a sample of size n, then the sample proportion defective is $p = d/n$. Hence, d is a binomial variate with parameters n and P.

$$ \therefore \quad E(d) = nP \quad \text{and} \quad \text{Var}(d) = nPQ, \quad Q = 1 - P $$

Thus $E(p) = E(d/n) = \dfrac{1}{n}E(d) = P$ and $\text{Var}(p) = \text{Var}(d/n) = \dfrac{1}{n^2}\text{Var}(d) = \dfrac{PQ}{n}$ ...(1·7)

Thus, the 3-σ control limits for p-chart are given by :

$$ E(p) \pm 3\, S.E.\,(p) = P \pm 3\sqrt{PQ/n} = P \pm A\sqrt{PQ} \qquad \text{...(1·8)} $$

where $A = 3/\sqrt{n}$ has been tabulated for different values of n.

Case (i) *Standards specified.* If P' is the given or known value of P, then

$$UCL_p = P' + A\sqrt{P'(1-P')} \quad ; \quad LCL_p = P' - A\sqrt{P'(1-P')} \quad ; \quad CL_p = P' \qquad \ldots(1\cdot8a)$$

Case (ii) *Standards not specified.* Let d_i be the number of defectives and p_i the fraction defective for the ith sample ($i = 1, 2, \ldots, k$) of size n_i. Then the population proportion p is estimated by the statistic $\bar{p}$ given by :

$$\bar{p} = \frac{\Sigma d_i}{\Sigma n_i} = \frac{\sum\limits_i n_i p_i}{\sum\limits_i n_i} \qquad \ldots(1\cdot8b)$$

It may be remarked here that $\bar{p}$ is an unbiased estimate of P, since

$$E(\bar{p}) = \sum_i E(d_i)/\Sigma n_i = \left[\Sigma(n_i P)/\Sigma n_i\right] = P$$

In this case

$$UCL_p = \bar{p} + A\sqrt{\bar{p}(1-\bar{p})} \quad ; \quad LCL_p = \bar{p} - A\sqrt{\bar{p}(1-\bar{p})} \quad ; \quad CL_p = \bar{p} \qquad \ldots(1\cdot8c)$$

1·9·2. Control Chart for Number of Defectives (*d*-chart). If instead of p, the sample proportion defective, we use d, the number of defectives in the sample, then the 3-σ control limits for d-chart are given by :

$$E(d) \pm 3\ S.E.\ (d) = nP \pm 3\sqrt{nP(1-P)} \qquad \ldots(1\cdot9)$$

Case (i) ***Standards specified.*** If P' is the given value of P then

$$UCL_d = nP' + 3\sqrt{nP'(1-P')} \quad ; \quad LCL_d = nP' - 3\sqrt{nP'(1-P')} \quad ; \quad CL_d = nP' \qquad \ldots(1\cdot9a)$$

Case (ii) ***Standards not specified.*** Using $\bar{p}$ as an estimate of P as in (1·8b), we get

$$UCL_d = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})} \quad ; \quad LCL_d = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})} \quad ; \quad CL_d = n\bar{p} \qquad \ldots(1\cdot9b)$$

Since p cannot be negative, if LCL as given by above formulae comes out to be negative, then it is taken to be zero.

***Remarks* 1. *p* and *d*-charts for Fixed Sample Size.** If the sample size remains constant for each sample *i.e.*, if $n_1 = n_2 = \ldots = n_k = n$, (say), then using (1·8$b$) an estimate of the population proportion p is given by :

$$\hat{p} = \bar{p} = \frac{\sum\limits_{i=1}^{k} d_i}{\sum\limits_i n_i} = \frac{\sum\limits_i d_i}{nk} = \frac{n\sum\limits_{i=1}^{k} p_i}{nk} = \frac{1}{k}\sum_{i=1}^{k} p_i \qquad \ldots(1\cdot9c)$$

In this case, the same set of control limits can be used for all the samples inspected and it is immaterial if one uses p-chart or d-chart.

2. *p* and *d*-charts for Variable Sample Size. *Method I.* If the number of items inspected (n) in each sample varies, for p-chart separate control limits have to be computed for each sample while the central line is invariant whereas for d-chart control limits as well as the central line has to be computed for each sample. This type of limits are known as *variable control limits*. In such a situation p-chart is relatively simple and is preferred to d-chart which becomes very confusing.

***Method* 2.** As pointed out in *Remark* 2, if n varies, separate control limits are calculated for each sample. Since S.E. $(p) = \sqrt{PQ/n}$, it should be noted that smaller the sample size wider the control band and *vice versa*. If the sample size does not vary appreciably then single set of control limits based on the average sample size $\left(\sum_{i=1}^{k} n_i / k\right)$ can be used. For practical purposes, this holds good for situations in which the largest sample size does not exceed the smallest sample size by more than 20% of the smallest sample size.

Alternatively, for all sample sizes two sets of limits, one based on the largest sample size and the other based on the smallest sample size can be used. The largest sample size gives the smallest control band which is called *inner band* and the smallest sample size gives the largest control band which is called *outer band*. Points falling within the inner band indicate the process in control while points lying outside the outer band are indicative of the presence of assignable causes of variation which must be searched and rectified. For other points, action should be based on the exact control limits.

***Method* 3.** Another procedure is to standardise the variate, *i.e.*, instead of plotting p or d on the control chart, we plot the corresponding standardised values, *viz.*,

$$Z = \frac{p - p'}{\sqrt{P'Q'/n}} \quad \text{or} \quad \frac{p - \bar{p}}{\sqrt{\bar{p}\,(1-\bar{p})/n}} \qquad \ldots(1{\cdot}10)$$

according as P is given or not, the symbols having their usual meanings. This stabilises our variable and the resulting chart is called *stabilised* p-chart or d-chart. In this case the control limits as well as the central line for p and d-charts are invariant with n (*i.e.*, they are constants independent of n) being given by :

$$UCL = 3, \qquad CL = 0, \qquad LCL = -3 \qquad \ldots(1{\cdot}10a)$$

Hence, the problem of variable control limits can be solved with a little more computational work discussed above.

Interpretations of p-chart. **1.** From the p-chart a process is judged to be in statistical control in the same way as is done for $\overline{X}$ and R charts. If all the sample points fall within the control limits without exhibiting any specific pattern, the process is said to be in control. In such a case, the observed variations in the fraction defective are attributed to the stable pattern of chance causes and the average fraction defective $\bar{p}$ is taken as the standard fraction defective P.

2. Points outside the UCL are termed as *high spots*. These suggest deterioration in the quality and should be regularly reported to the production engineers. The reasons for such deterioration could possibly be known and removed if the details of conditions under which data were collected, were known. Of particular interest and importance is, if there was any change of inspection or inspection standards.

3. Points below LCL are called *low spots*. Such points represent a situation showing improvement in the product quality. However, before taking this improvement for guaranteed, it should be investigated if there was any slackness in inspection or not.

4. When a number of points fall outside the control limits, a revised estimate of P should be obtained by eliminating all the points that fall above UCL (it is assumed that points that fall below LCL are not due to faulty inspection). The standard fraction defective P should be revised periodically in this way.

Remark. The interpretation for the control chart for number of defects (d-chart) is same as that for p-chart.

Example 1·10. *The following are the figures of defectives in 22 lots each containing 2,000 rubber belts :*

425, 430, 216, 341, 225, 322, 280, 306, 337, 305, 356,
402, 216, 264, 126, 409, 193, 326, 280, 389, 451, 420

Draw control chart for fraction defective and comment on the state of control of the process.

Solution. Here we have a fixed sample size $n = 2{,}000$ for each lot. If d_i and p_i are respectively the number of defectives and the sample fraction defective for the ith lot, then

$$p_i = \frac{d_i}{2{,}000}, \ (i = 1, 2, \ldots, 22)$$

which are given in Table 1·2.

TABLE 1·2 : COMPUTATIONS FOR C.C. FOR FRACTION DEFECTIVE

S. No.	d	p = (d/2000)	S. No.	d	p = (d/2000)
1	425	0·2125	12	402	0·2010
2	430	0·2150	13	216	0·1080
3	216	0·1080	14	264	0·1320
4	341	0·1705	15	126	0·0630
5	225	0·1125	16	409	0·2045
6	322	0·1610	17	193	0·0965
7	280	0·1400	18	326	0·1630
8	306	0·1530	19	280	0·1400
9	337	0·1685	20	389	0·1945
10	305	0·1525	21	451	0·2255
11	356	0·1780	22	420	0·2100
Total	3,543	1·7715		3,476	1·7380

In the usual notations, we have

$$\bar{p} = \frac{\Sigma p_i}{k} = \frac{1{\cdot}7715 + 1{\cdot}7380}{22} = \frac{3{\cdot}5095}{22} = 0{\cdot}1595 \quad \Rightarrow \quad \bar{q} = 1 - \bar{p} = 0{\cdot}8405$$

$$\left[\text{Or} \quad \bar{p} = \frac{\Sigma d_i}{nk} = \frac{3543 + 3476}{2000 \times 22} = \frac{7019}{44000} = 0{\cdot}1595\right]$$

3-σ control limits for p-chart are given by :

$$\bar{p} \pm 3\sqrt{\bar{p}\,\bar{q}/n} = 0{\cdot}1595 \pm 3\sqrt{0{\cdot}1595 \times 0{\cdot}8405/2000}$$

$$= 0{\cdot}1595 \pm 3\sqrt{0{\cdot}000067} = 0{\cdot}1595 \pm 0{\cdot}0246$$

$$\therefore \quad UCL_p = 0{\cdot}1595 + 0{\cdot}0246 = 0{\cdot}1841; \ LCL_p = 0{\cdot}1595 - 0{\cdot}0246 = 0{\cdot}1349; \ CL_p = \bar{p} = 0{\cdot}1595$$

The control chart for fraction defective (p-chart) is drawn in *Fig. 1·9*.

From the p-chart, we find that the sample points (fraction defectives) corresponding to the sample numbers 1, 2, 3, 5, 12, 13, 14, 15, 16, 17, 20, 21 and 22, fall outside the control limits. Hence, the process cannot be regarded in statistical control.

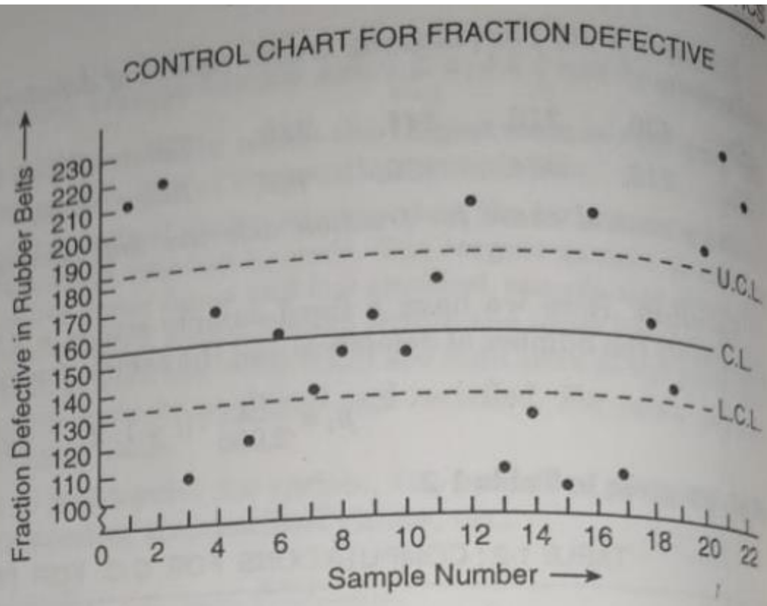

Fig. 1·9

Example 1·11. *From the following inspection results, construct 3-σ control limits for p chart :*

Date Sept.	No. of Defectives	Date Sept.	No. of Defectives	Date Sept.	No. of Defectives
1	22	11	70	21	66
2	40	12	80	22	50
3	36	13	44	23	46
4	32	14	22	24	32
5	42	15	32	25	42
6	40	16	42	26	46
7	30	17	20	27	30
8	44	18	46	28	38
9	42	19	28	29	40
10	38	20	36	30	24

The sub-groups, from which the defectives were taken out, were of the same size, i.e., 1,000 items each.

Without constructing the control chart, comment on the state of control of the process. If the process is out of control, then suggest the revised control limits for future use.

Solution. Here we have a fixed sample size for each lot. If d_i and p_i are respectively the number of defectives and the sample fraction defective for the ith lot then

$$p_i = \frac{d_i}{1{,}000}, \quad (i = 1, 2, \ldots, 30)$$

which are given in *Table 1·3* :

TABLE 1·3 : CALCULATIONS FOR CONTROL LIMITS FOR p-CHART

Date Sept.	*No. of defectives*	*Fraction defectives*	*Date Sept.*	*No. of defectives*	*Fraction defective*	*Date Sept.*	*No. of defectives*	*Fraction defectives*
1	22	0·022	11	70	0·070	21	66	0·066
2	40	0·040	12	80	0·080	22	50	0·050
3	36	0·036	13	44	0·044	23	46	0·046
4	32	0·032	14	22	0·022	24	32	0·032
5	42	0·042	15	32	0·032	25	42	0·042
6	40	0·040	16	42	0·042	26	46	0·046
7	30	0·030	17	20	0·020	27	30	0·030
8	44	0·044	18	46	0·046	28	38	0·038
9	42	0·042	19	28	0·028	29	40	0·040
10	38	0·038	20	36	0·036	30	24	0·024
Total	366	0·366	Total	420	0·420	Total	414	0·414

From the above table, we have

$\sum d_i = 366 + 420 + 414 = 1{,}200$; $\sum p_i = 0{\cdot}366 + 0{\cdot}420 + 0{\cdot}414 = 1{\cdot}200$; $n = 1000,\ k = 30$

$$\bar{p} = \frac{\sum d}{nk} = \frac{1200}{1000 \times 30} = 0{\cdot}040 \quad \text{or} \quad \bar{p} = \frac{\sum p_i}{k} = \frac{1{\cdot}2}{30} = 0{\cdot}040$$

3-σ Control Limits for p-Chart : $CL_p = \bar{p} = 0{\cdot}040$

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0{\cdot}040 + 3\sqrt{\frac{0{\cdot}04(1-0{\cdot}04)}{1000}}$$

$$= 0{\cdot}040 + 3\sqrt{0{\cdot}0000384} = 0{\cdot}040 + 3 \times 0{\cdot}0062 = 0{\cdot}0586$$

$$LCL_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0{\cdot}040 - 0{\cdot}0186 = 0{\cdot}0214.$$

We observe that the sample points (fraction defectives) on 11th, 12th, 17th and 21st September were 0·070, 0·080, 0·020 and 0·066 respectively and these fall outside the control limits. Hence the process is not in a state of statistical control.

Revised Control Limits : The revised control limits are obtained on eliminating these four samples and considering the remaining 30 – 4 = 26 samples.

Based on the remaining 26 samples, we get

$$CL_p = \bar{p}' = \frac{\sum d - 70 - 80 - 20 - 66}{1000 \times 26} = \frac{1200 - 236}{1000 \times 26} = \frac{964}{26000} = 0{\cdot}0371$$

or

$$CL_p = \bar{p}' = \frac{\sum p - 0{\cdot}07 - 0.08 - 0.02 - 0.066}{26}$$

$$= \frac{1{\cdot}200 - 0{\cdot}236}{26} = \frac{0{\cdot}964}{26} = 0{\cdot}0371$$

$$UCL_p = \bar{p}' + 3\sqrt{\frac{\bar{p}'\bar{q}'}{n}} = 0{\cdot}0371 + 3\sqrt{\frac{0{\cdot}0371 \times 0{\cdot}9629}{1000}}$$

$$= 0{\cdot}0371 + 3 \times \sqrt{0{\cdot}0000357} = 0{\cdot}0371 + 3 \times 0{\cdot}0060$$

$$= 0{\cdot}0371 + 0{\cdot}0180 = 0{\cdot}0551$$

$$LCL_p = \bar{p}' - 3\sqrt{\frac{\bar{p}'\bar{q}'}{n}} = 0{\cdot}0371 - 0{\cdot}0180 = 0{\cdot}0191$$

Since none of the remaining 26 sample points (fraction defectives) lies outside the revised control limits, $[LCL_p = 0{\cdot}0191$ and $UCL_p = 0{\cdot}0551]$, these may be regarded as the control limits for p-chart for the future production from this process.

Example 1·12. *20 samples each of size 10 were inspected. The number of defectives detected in each of them is given below :*

Samples No.	:	1	2	3	4	5	6	7	8	9	10
No. of defectives	:	0	1	0	3	9	2	0	7	0	1
Sample No.	:	11	12	13	14	15	16	17	18	19	20
No. of defectives	:	1	0	0	3	1	0	0	2	1	0

Construct the 'number of defectives' chart and establish quality standard for the future.

Solution. Here we have samples of fixed size $n = 10$. The total number of defectives in all the 20 samples is :

$$\Sigma d = 0 + 1 + 0 + 3 + 9 + \ldots + 2 + 1 + 0 = 31$$

An estimate of the process fraction defective is given by :

$$\bar{p} = \frac{\Sigma d}{nk} = \frac{31}{10 \times 20} = 0{\cdot}155 \quad \Rightarrow \quad \bar{q} = 1 - \bar{p} = 0{\cdot}845$$

The 3-σ control limits for 'number defectives' chart (np-chart) are given by :

$$CL_{np} = n\bar{p} = 1{\cdot}55$$

$$UCL_{np} = n\bar{p} + 3\sqrt{n\bar{p}\,(1-\bar{p})} = 1{\cdot}55 + 3\sqrt{1{\cdot}55 \times 0{\cdot}845} = 1{\cdot}55 + 3{\cdot}43 = 4{\cdot}98$$

$$LCL_{np} = n\bar{p} - 3\sqrt{n\bar{p}\,(1-\bar{p})} = 1{\cdot}55 - 3\sqrt{1{\cdot}55 \times 0{\cdot}845} \simeq 0 \text{ (Negative)}$$

The control chart for the 'number of defectives' is obtained on plotting the number of defectives against the corresponding sample number and is given in *Fig. 1·10.*

***np*-Chart**

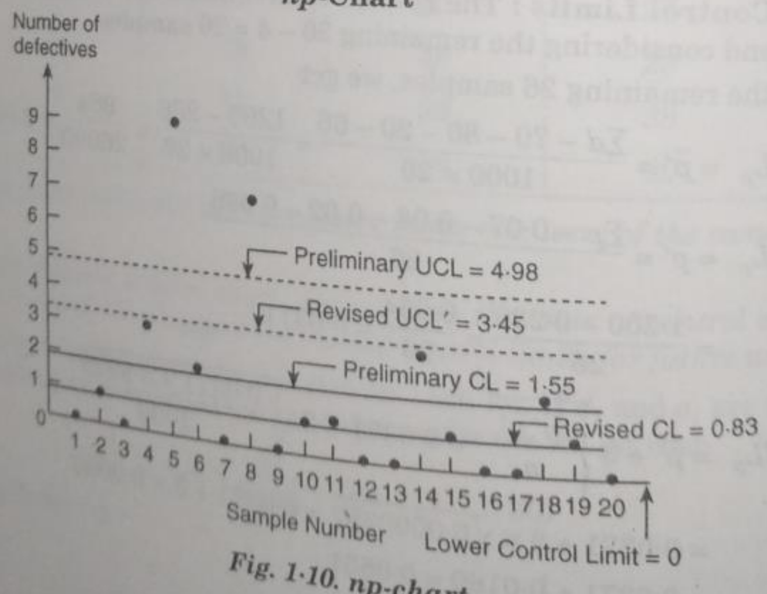

Fig. 1·10. np-chart.

Since two points corresponding to 5th and 8th samples lie outside the control limits, we conclude that the process is not in a state of statistical control. To establish quality standards for the future, we eliminate these points.

Deleting sample numbers 5 and 8, we compute *CL*, *UCL* and *LCL* based on the remaining 20 – 2 = 18 samples as follows :

Revised Control Limits for Future :

$$\bar{p}' = \frac{\Sigma d - 9 - 7}{10 \times 18} = \frac{31 - 16}{180} = \frac{15}{180} = 0{\cdot}083 \qquad \Rightarrow \qquad \bar{q}' = 1 - \bar{p}' = 0{\cdot}917$$

$$CL' = n\bar{p}' = 10 \times 0{\cdot}083 = 0{\cdot}83$$

$$UCL' = n\bar{p}' + 3\sqrt{n\bar{p}'(1-\bar{p}')} = 0{\cdot}83 + 3\sqrt{0{\cdot}83 \times 0{\cdot}917} = 0{\cdot}83 + 2{\cdot}62 = 3{\cdot}45$$

$$LCL' = np' - 3\sqrt{n\bar{p}'(1-\bar{p}')} = 0{\cdot}38 - 3\sqrt{0{\cdot}83 \times 0{\cdot}917} = -0{\cdot}178 \simeq 0.$$

These revised values are shown in the (on page 1.36) as the revised *UCL'* and *LCL'* respectively. No sample values other than those which have been deleted, fall outside the new limits. We take these new limits, alongwith the new central line, as standards for controlling product in the future.

Example 1·13. *The following data give the number of defectives in 10 independent samples of varying sizes from a production process :*

Sample No. :	1	2	3	4	5	6	7	8	9	10
Sample size :	2,000	1,500	1,400	1,350	1,250	1,760	1,875	1,955	3,125	1,575
No. of defectives :	425	430	216	341	225	322	280	306	337	305

Draw the control chart for fraction defective and comment on it.

Solution. Since we have variable sample size, we can draw the control chart for fraction defective in the following three ways.

Method 1. *Variable Control Limits.* In this case we calculate 3-σ limits for each sample separately by using the formula :

$$UCL = \bar{p} + 3\sqrt{\bar{p}\,\bar{q}/n_i} \quad \text{and} \quad LCL = \bar{p} - 3\sqrt{\bar{p}\,\bar{q}/n_i}, \quad \text{where} \quad \bar{p} = \frac{\Sigma d_i}{\Sigma n_i} = \frac{3{,}187}{17{,}790} = 0{\cdot}1791 \ldots (*)$$

and d_i = No. of defectives in the *i*th sample, n_i = sample size of the *i*th sample

∴ $\bar{q} = 1 - \bar{p} = 0{\cdot}8209$. Thus, $\bar{p}\,\bar{q} = 0{\cdot}1791 \times 0{\cdot}8209 = 0{\cdot}1470231$

For computation of the variable control limits, see the calculation in *Table 1·4*.

TABLE 1·4 : COMPUTATIONS FOR *p*-CHART (VARIABLE CONTROL LIMITS)

n	d	$p = d/n$	$\bar{p}\bar{q}/n$	$\sqrt{\bar{p}\,\bar{q}/n}$	$3 \times \sqrt{\bar{p}\,\bar{q}/n}$	*UCL*	*LCL*
2,000	425	·2125	·0000735	·008573	·025719	0.205	0.153
1,500	430	·2867	·000098	·009899	·029698	0.209	0.149
1,400	216	·1543	·000105	·010247	·030741	0.210	0.148
1,350	341	·2526	·000109	·010440	·031321	0.210	0.148
1,250	225	·1800	·000118	·010863	·032588	0.212	0.147

1,760	322	·1829	·000084	·009138	·027413	.207	.152
1,875	280	·1495	·000078	·008854	·026562	.206	.153
1,955	306	·1565	·000075	·008672	·026015	.205	.153
3,125	337	·1078	·000047	·006856	·020567	.200	.159
1,575	305	·1937	·000093	·009659	·028977	.208	.150
17,790	3,187						

From the chart [Fig. 1·11], it is obvious that a number of sample points corresponding to sample numbers 1, 2, 4, 7 and 9 are outside the respective control limits. Hence the process is not in a state of statistical control. This suggests the presence of some assignable causes of variations, which should be detected and eliminated.

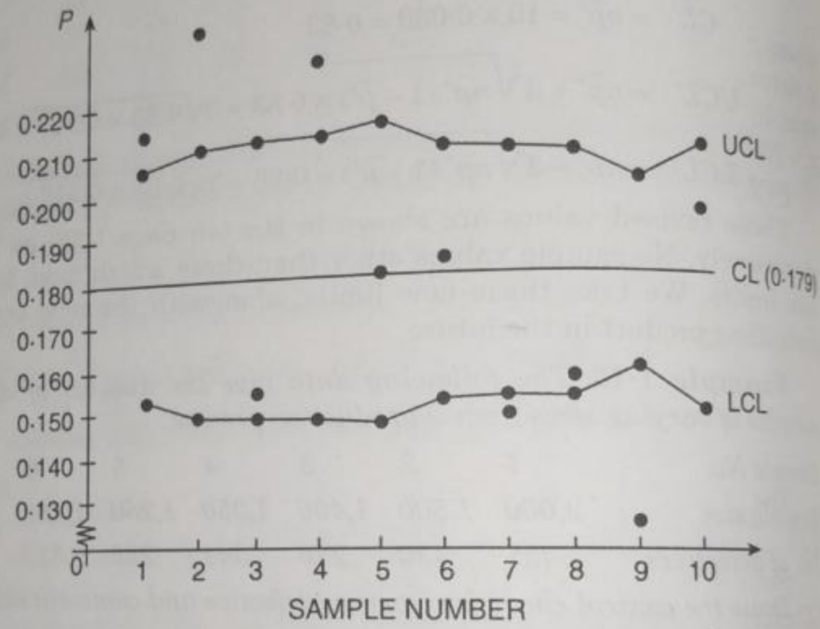

Fig. 1·11

Method 2. Here we set up two sets of control limits, one based on the maximum sample size n = 3,125 (corresponding to 9th sample) and the other based on the minimum sample size n = 1,250 (corresponding to 5th sample). From the table, we note that the corresponding sets of control limits are :

For $n = 3{,}125$; $UCL = 0{\cdot}1997$ and $LCL = 0{\cdot}1585$; For n = 1,250; $UCL = 0{\cdot}2117$ and $LCL = 0{\cdot}1465$.

From the control chart [*Fig. 1·12*], we find that the sample points corresponding to sample number 1, 2, 4 and 9 lie outside

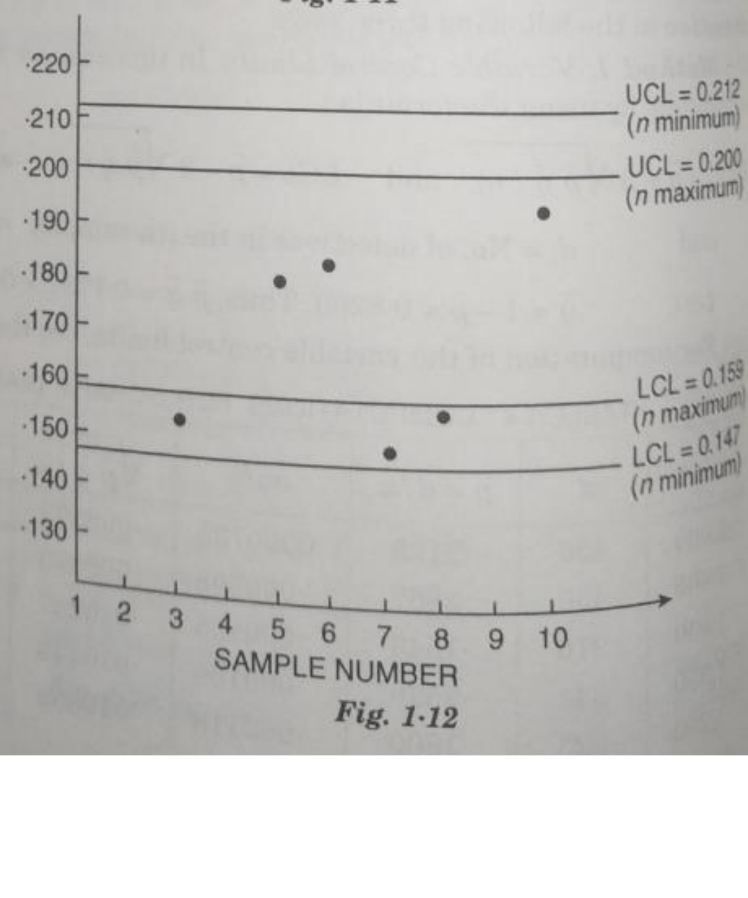

Fig. 1·12

the outer band (based on minimum sample size), and hence the process is out of statistical control.

Method 3. We standardise the statistic p by the formula :

$$Z_i = \frac{p_i - E(p_i)}{S.E.\ (p_i)} = \frac{p_i - \bar{p}}{\sqrt{\bar{p}\bar{q}/n}} \ , \qquad \ldots(*)$$

where $\bar{p}$ is computed in (*) and plot the Z-values against the corresponding sample number. Since n is large, $Z_i \sim N(0, 1)$ and hence

$UCL_z = 3$; $LCL_z = -3$; $CL_z = 0$

TABLE 1·5 : COMPUTATIONS OF Z-VALUES

p	$p - \bar{p}$	$\sqrt{\bar{p}\,\bar{q}\,/\,n}$	$Z = \dfrac{p-\bar{p}}{\sqrt{\bar{p}\,\bar{q}/n}}$
·2126	·0334	·0086	3·8841
·2867	·1076	·0099	10·8686
·1543	–·0248	·0102	–2·4313
·2526	·0735	·0104	5·25
·1800	·0009	·0109	0·0826
·1829	·0038	·0091	0·4176
·1495	–·0296	·0089	–3·2558
·1565	–·0226	·0087	–2·5977
·1078	–·0713	·0069	–10·3333
·1937	·0146	·0097	1·5052

The control chart is drawn in *Fig.* 1·13.

Since a number of sample values (Z) except for sample numbers 3, 5, 6, 8 and 10 lie-outside the limits ± 3, therefore the process is out of statistical control.

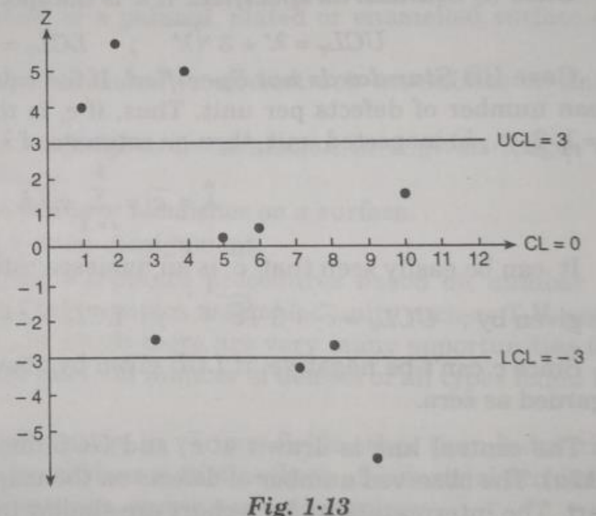

Fig. 1·13

1·9·3. Control Chart for Number of Defects per Unit (c - Chart). The field of application of c-chart is much more restricted as compared to $\overline{X}$ and R charts or p-chart. Before we embark upon to discuss the theory behind c-chart, it is imperative to distinguish between defect and detective. An article which does not conform to one or more of the specifications, is termed as *defective* while any instance of article's lack of conformity to specifications is a *defect.* Thus, every defective contains one or more of the defects, *e.g.*, a defective casting may further be examined for blow holes, cold shuts, rough surface, weak structure, etc.

Unlike d or np-chart which applies to the number of defectives in a sample, c-chart applies to the number of defects per unit. Sample size for c-chart may be a single unit like a radio, or group of units or it may be a unit of fixed time, length, area, etc. For example, in case of surface defects, area of the surface is the sample size ; in case of casting defects, a single part (such as base plate, side cover) is the sample size. However, defined sample size should be constant in the sense that different samples have essentially equal opportunity for the occurrence of defects.

Control Limits for *c*-chart. In many manufacturing or inspection situations, the sample size n *i.e.*, the area of opportunity is very large (since the opportunities for defects to occur are numerous) and the probability p of the occurrence of a defect in any one spot is very small such that np is finite. In such situations from statistical theory we know that the pattern of variations in data can be represented by Poisson distribution, and consequently 3-σ control limits based on Poisson distribution are used. Since for a Poisson distribution, mean and variance are equal, if we assume that c is Poisson variate with parameter, λ, we get

$$E(c) = \lambda \quad \text{and} \quad \text{Var}(c) = \lambda$$

Thus 3-σ control limits for c-chart are given by :

$$\left.\begin{aligned} UCL_c &= E(c) + 3\sqrt{\text{Var}(c)} = \lambda + 3\sqrt{\lambda} \\ LCL_c &= E(c) - 3\sqrt{\text{Var}(c)} = \lambda - 3\sqrt{\lambda} \\ CL_c &= \lambda \end{aligned}\right\} \quad ...(1\cdot11)$$

Case (*i*) *Standards specified.* If λ' is the specified value of λ, then

$$UCL_c = \lambda' + 3\sqrt{\lambda'} \quad ; \quad LCL_c = \lambda' - 3\sqrt{\lambda'} \quad ; \quad CL_c = \lambda' \quad ...(1\cdot12)$$

Case (*ii*) *Standards not Specified.* If the value of λ is not known, it is estimated by the mean number of defects per unit. Thus, if c_i is the number of defects observed on the ith $(i = 1, 2, ..., k)$ inspected unit, then an estimate of λ is given by :

$$\hat{\lambda} = \bar{c} = \sum_{i=1}^{k} c_i / k \quad ...(1\cdot12a)$$

It can be easily seen that $\bar{c}$ is an unbiased estimate of λ. The control limits, in this case, are given by : $UCL_c = \bar{c} + 3\sqrt{\bar{c}}$; $LCL_c = \bar{c} - 3\sqrt{\bar{c}}$; $CL_c = \bar{c}$ ...(1·12b)

Since c can't be negative, if LCL given by above formulae comes out to be negative, it is regarded as zero.

The central line is drawn at $\bar{c}$, and UCL and LCL are drawn at the values given by (1·12a). The observed number of defects on the inspected units are then plotted on the control chart. The interpretations for c-chart are similar to those of p-chart.

Remark. Usually k, the number of samples (inspected units), is taken from 20 to 25. Normal approximation to Poisson distribution may be used provided $\bar{c} < 5$.

1·9·4. *c*- Chart for Variable Sample Size or *u*-Chart. In this case instead of plotting c, the statistic $u = c / n$ is plotted, n being the sample size which is varying. If n_i is the sample size and c_i the total number of defects observed in the ith sample, then

$$u_i = c_i / n_i, \ (i = 1, 2, ..., k), \quad ...(1\cdot13)$$

gives the average number of defects per unit for the ith sample.

In this case an estimate of λ, the mean number of defects per unit in the lot, based on all the k-samples is given by :

$$\hat{\lambda} = \bar{u} = \frac{1}{k} \sum_{i=1}^{k} u_i \quad ...(1\cdot13a)$$

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We know that if $\overline{X}$ is the mean of a random sample of size n then S.E. $(\overline{X}) = \sigma/\sqrt{n}$. Hence, the standard error of the average number of defects per unit is given by :

$$S.E.\ (u) = \sqrt{\lambda / n} = \sqrt{\overline{u} / n} \quad ; \quad \text{[On using (1·13}a\text{)]} \qquad \text{...(1·13}b\text{)}$$

Hence, 3-σ countrol limits for u-chart (or c - Chart for variable sample size) are given by :

$$\text{UCL}_u = \overline{u} + 3\sqrt{\overline{u}/n} \quad ; \quad \text{LCL}_u = \overline{u} - 3\sqrt{\overline{u}/n} \quad ; \quad CL_u = \overline{u} \qquad \text{... (1·13}c\text{)}$$

As is obvious, control limits will vary for each sample. The central line, however, will be same. The interpretation of these charts is similar to the p-chart or d-chart.

Applications of *c*-chart

The universal nature of Poisson distribution as the law of small numbers makes the c-chart technique quite useful. In spite of the limited field of application of c-chart (as compared to $\overline{X}$, R, p-charts), there do exist situations in industry where c-chart is definitely needed. Some of the representative types of defects to which c-chart can be applied with advantage are :

1. c is number of imperfections observed in a bale of cloth.

2. c is the number of surface defects observed in (*i*) roll of coated paper or a sheet of photographic film, and (*ii*) a galvanised sheet or a painted, plated or enamelled surface of given area.

3. c is the number of defects of all types observed in aircraft sub-assemblies or final assembly.

4. c is the number of breakdowns at weak spots in insulation in a given length of insulated wire subject to a specified test voltage.

5. c is the number of defects observed in stains or blemishes on a surface.

6. c is the number of soiled packages in a given consignment.

7. c-chart has been applied to sampling acceptance procedures based on number of defects per unit, *e.g.*, in case of inspection of fairly complex assembled units such as T.V. sets, aircraft engines, tanks, machine-guns, etc., in which there are very many opportunities for the occurrence of defects of various types and the total number of defects of all types found by inspection is recorded for each unit.

8. c-chart technique can be used with advantage in various fields other than industrial quality control, *e.g.*, it has been applied (*i*) to accident statistics (both of industrial accidents and highway accidents), (*ii*) in chemical laboratories, and (*iii*) in epidemiology.

Example 1·14. *In welding of seams, defects included pinholes, cracks, cold taps, etc. A record was made of the number of defects found in one seam each hour and is given below.*

1·12·2005	*8 A.M.*	*2*	*2·12·2005*	*8 A.M.*	*5*	*3·12·2005*	*8 A.M.*	*6*
	9 A.M.	*4*		*9 A.M.*	*3*		*9 A.M.*	*4*
	10 A.M.	*7*		*10 A.M.*	*7*		*10 A.M.*	*3*
	11 A.M.	*3*		*11 A.M.*	*11*		*11 A.M.*	*9*
	12 A.M.	*1*		*12 A.M.*	*6*		*12 A.M.*	*7*
	1 P.M.	*4*		*1 P.M.*	*4*		*1 P.M.*	*4*
	2 P.M.	*8*		*2 P.M.*	*9*		*2 P.M.*	*7*
	3 P.M.	*9*		*3 P.M.*	*9*		*3 P.M.*	*12*

Draw the control chart for number of defects and give your comments.

Solution. Average number of defects per sample is : $\bar{c} = \frac{1}{k}\Sigma c = \frac{1}{24} \times 144 = 6$

The control limits and the central line, therefore are as follows :

$UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 6 + 3\sqrt{6} = 13{\cdot}35$

$LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 6 - 3\sqrt{6} = -1{\cdot}35$

$CL_c = \bar{c} = 6$

Because the number of defects cannot be negative, so we consider the lower limit to be zero, *i.e.*, $\bar{c}$ is allowed to vary between 0 and 13·35.

The control chart is drawn in *Fig. 1·14*.

Since none of the 24 points falls outside the control limits, process average may be regarded in state of statistical control.

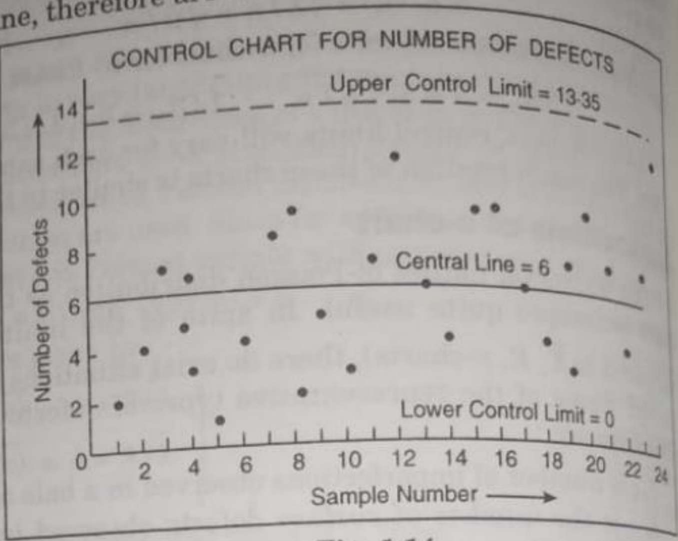

Fig. 1·14

Example 1·15. *The number of defects on 20 items are given below :*

Item No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of defects	2	0	4	1	0	8	0	1	2	0	6	0	2	1	0	3	2	1	0	2

Devise a suitable control scheme for the future.

Solution. The control chart to be used is the *c*-chart.

$$\bar{c} = \text{Average number of defects/item} = \frac{1}{k}\Sigma c = \frac{1}{20} \times 35 = 1{\cdot}75$$

$CL_c = \bar{c} = 1{\cdot}75$; $UCL_c = \bar{c} + 3\sqrt{\bar{c}} = 1{\cdot}75 + 3\sqrt{1{\cdot}75} = 5{\cdot}71$; $LCL_c = \bar{c} - 3\sqrt{\bar{c}} = 1{\cdot}75 - 3{\cdot}96 \simeq 0$

Since two sample points corresponding to 6th and 11th samples lie outside the control limits, we conclude that the process is not in a state of statistical control. To establish quality standards for the future, we eliminate these 'out of control' sample points. Deleting sample numbers 6 and 11, we compute the new control limits CL', UCL' and LCL' for the remaining 18 samples as follows :

$$\bar{c}' = \frac{\Sigma c - 8 - 6}{20 - 2} = \frac{35 - 14}{18} = \frac{21}{18} = 1{\cdot}17$$

$CL' = \bar{c}' = 1{\cdot}17$; $UCL' = \bar{c}' + 3\sqrt{\bar{c}'} = 1{\cdot}17 + 3\sqrt{1{\cdot}17} = 4{\cdot}15$;

$LCL' = \bar{c}' - 3\sqrt{\bar{c}'} = 1{\cdot}17 - 3\sqrt{1{\cdot}17} \simeq 0$

It may be noted that now no sample points (c-values) other than those which have been deleted, fall outside the new control limits. We take these new limits, alongwith the new central line, as standards for controlling production in the future.

1·10. NATURAL TOLERANCE LIMITS AND SPECIFICATION LIMITS

A process in statistical control implies that the control charts for both the mean and range show complete homogeneity and in such a case, a measure of the variation of the individual products is given by the standard deviation (σ), estimate by $\bar{R}/d_2$ or $\bar{s}/c_2$ from control data. If μ and σ are the process average and process standard deviation respectively, then the limits $\mu \pm 3\sigma$ are called the *Natural Tolerance Limits,*. The probability of an observation lying outside these limits is 0·0027. The width '6σ' which is the inherent variability of the process is given a special name *Natural Tolerance*. If μ and σ are not known then $\hat{\mu} \pm 3\hat{\sigma}$ are the estimates of the natural tolerance limits where

$$\hat{\mu} = \overline{\overline{X}} \quad \text{and} \quad \hat{\sigma} = \overline{R}/d_2 \quad \text{or} \quad \hat{\sigma} = \overline{s}/C_2.$$

It might happen that even though the process is in statistical control as exhibited by control charts, the customer may not be satisfied with the product. This happens when the process does not conform to *specification limits* (limits as desired or fixed by the customer) for that item. These specification limits are generally given in terms of upper and lower tolerance limits. A dicision, whether a process needs adjustment or not, can be made at the point by comparing natural tolerance limits and specification limits.

Comparison. Let X_{mas} and X_{min} denote the upper specification limit (*U.S.L.*) and lower specification limit (*L.S.L.*) respectively for some quality characteristic. When both these limits are specified, a comparison of these with the 'natural tolerance limits' may result in one of the following three situations :

(*a*) *Natural tolerance is considerably smaller than specified tolerance, i.e.,* $X_{max} - X_{min} > 6\sigma$.

Interpretations. (*i*) In such a case almost all the manufactured items will conform to specifications as long as the process is in statistical control and is appropriately centered as in positions *A, B* or *C* as shown in the *Fig. 1·15.*

NATURAL TOLERANCE SMALLER THAN SPECIFIED TOLERANCE

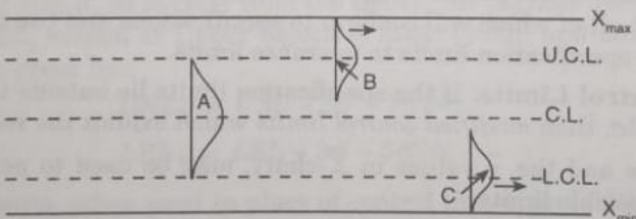

Fig. 1·15

If the process is operating under one of these conditions, $\overline{X}$ may be permitted to go out of control, provided it does not go too far ; in other words, the distribution of $\overline{X}$ may be allowed to fluctuate between positions *B* and *C*. This will save the time and money for frequent machine setting and delays due to looking for assignable causes of variation which will not be responsible for unsatisfactory product.

(*ii*) In such a situation since, even considerable shifts in the level of working may not result in the items falling outside specification limits, the time interval between taking successive samples for control chart inspection can be appreciably increased.

(*iii*) The larger the ratio ($X_{max} - X_{min}$) to the natural tolerance 6σ, the greater is the likelihood of getting good product without assistance from any control chart. This will imply that the process is too good for the product and it may be economical to examine if relaxations in the conditions of production, *e.g.,* less costly experiment or processing or material, could be allowed. It may also be worthwhile to '*squeeze*' the specification limits, to produce a product superior to the one originally intended.

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(*b*) *Specification limits coincide with tolerance limits, i.e.,*

$$X_{max} - X_{min} \simeq 6\sigma$$

This is an ideal situation and in this case a process in statistical control obviously implies that the product is meeting the specifications. Here, careful centering of the process is all the more important and if no item is to be rejected then the process has to be centred exactly at the specification mean. Any departure from this centering would result in some of the product going outside the specification limits. As soon as a control chart detects such departure, immediate remedial action should be taken to maintain the centering of the process.

(*c*) *Natural tolerance is greater than specified tolerance, i.e.,*

$$X_{max} - X_{min} < 6\sigma$$

Interpretations. (*i*) If the natural tolerances are not included within the specification limits then even with the process in control and the process average perfectly centered at the specification mean, the production of an appreciable quantity of defective articles (*i.e.*, articles *not* conforming to specifications) is inevitable. Here a slight shift in the process average will increase the per cent defective. In such a situation, a re-adjustment of the process is advisable with respect to either the process average or process dispersion or both.

(*ii*) Also it would be worthwhile to investigate the possibility of relaxing the specified tolerances to the extent of natural tolerances.

If 100% inspection is possible, then defective articles may be sorted out and eliminated but if 100% inspection is not possible (*e.g.*, if testing is destructive) then there is no chance of getting the product all of which will conform to specifications and the only alternative in this case is to relax the specification limits to tolerance limits.

Modified Control Limits. If the specification limits lie outside the natural tolerances, *i.e.*, $X_{max} - X_{min} > 6\sigma$, then *modified control limits* which exhibit the relationship between the specification limits and the $\bar{x}$ values in $\overline{X}$-chart, may be used to permit shifts in process levels within permissible limits.

As already pointed out, in such a situation shifts in the values of $\overline{X}$ may be allowed provided it does not go too far. This poses the question : "What are the limits within which $\overline{X}$-values may be allowed to vary such that the product meets the specifications?", since if the process is centered at *A* and *B* as shown in the *Fig. 1·16a* some of the items will naturally lie outside the specification limits.

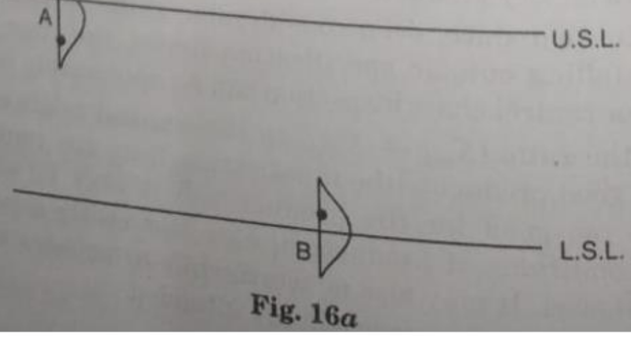

Fig. 16*a*

Let us have a look at the following figure (*Fig. 1·16b*) which shows the statistically controlled universe in its highest and lowest accepting positions.

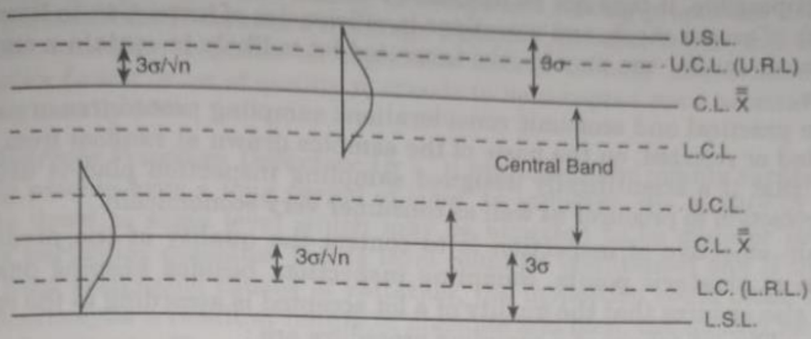

Fig. 1·16b

The natural tolerances (*i.e.*, process dispersion) is 6σ. If the universe is at the highest accepting position, then the process average (central line) will be at a distance 3-σ below USL and similarly when the universe is at its lowest accepting position, the process average is at a distance 3σ above the LSL. Thus in this case, instead of fixed central line at $\bar{\bar{X}}$, we have a central band so that as long as $\bar{X}$ lies in this central band, the product will conform to specifications. The upper and lower edges of the central band are given respectively by

$$USL - 3\sigma,\ LSL + 3\sigma$$

For a sub-group of size n, as is clear from the figure, the highest and lowest satisfactory values of UCL and LCL, known as Upper Rejection Limit (URL) and Lower Rejection Limit (LRL) respectively are given by :

$$URL_{\bar{X}} = USL - 3\sigma' + 3\sigma'/\sqrt{n}$$

$$LRL_{\bar{X}} = LSL + 3\sigma' - 3\sigma'/\sqrt{n}$$

These rejection limits, when used in place of control limits, are called *'modified control limits'*.

1·11. ACCEPTANCE SAMPLING INSPECTION PLANS

In many a manufacturing process, the producer, in order to ensure that the manufactured goods are according to specifications of the customer, gets his lot checked at strategic stages. On the other hand, the customer is anxious to satisfy himself about the quality of goods he accepts. An ideal way of doing this seems to inspect each and every item presented for acceptance, *i.e.*, to resort to 100 per cent inspection. 100% inspection should be taken recourse to under the following conditions :

(*i*) The occurrence of a defect may cause loss of life or serious casualty to personnel.

(*ii*) A defect may cause serious malfunction of equipment.

We may also wish to examine all the items of the product under the following conditions :

(*i*) N, the lot size is small, and

(*ii*) The incoming quality is poor or unknown.

Control charts

Variables
- Mean ($\bar{X}$)
- Range(R)
- Standard Deviation (σ)

Attributes
- Fraction defective (or) (P)
- Number of defective (or) (d) (or) np chart

Control charts for Attributes for
(a) Fixed Sample Size
(b) Variable Sample Size.

Remark – ① If any points fall outside the control limits then those points can be deleted and Revised control charts are constructed.
② P chart is preferred to that of d chart.

Control chart for Variable Sample Size

Procedure –
Method – I.
construct "σ" limits for each sample.

$$UCL = \bar{P} + 3.\sqrt{\frac{\bar{P}\bar{q}}{n}}$$ and

$$LCL = \bar{P} - 3\sqrt{\frac{\bar{P}\bar{q}}{n}}$$

where $\bar{P} = \frac{\Sigma d_i}{n_i}$

Method – II.
Construct "σ" limits, for
(i) Minimum "n"
(ii) Maximum "n".

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