Neymann factorization theorem

Let $\underset{\sim}{x} = (x_1, x_2 \ldots x_n)$ be a random sample with probability function $f(x, \theta)$ involving unknown parameter 'θ' then a statistic $T(x)$ is said to be sufficient for θ iff for every $\theta \in \Omega$.

$$f(x, \theta) = g(T(x), \theta) \cdot h(x).$$

where g is a non-negetive function of $T(x)$ and θ and h is a non-ve fun. of x alone free of θ

Proof:-

Necessary condition

Suppose $T(x)$ is sufficient for θ, the conditional $dist^n$. when the $P(X=x) > 0$ is independent of θ of x given $T(x) = t$

$$P(x, \theta) = P[X=x, T(x)=t]$$

$$= P[T(x)=t] \cdot P[X=x / T(x)=t]$$

$$= P[T(x), \theta] \cdot h(x).$$

$P(X=x) > 0, \ \theta \in \Omega$

for every x for which

Sufficient condition:-

Suppose $f(x, \theta) = g(T(x), \theta) \cdot h(x)$. holds and $P[T(x) = t_0] > 0$ for fixed $T(x) = t_0$ and every $\theta \in \Omega$.

By setting $B_0 = \{x : T(x) = t_0\}$

$$P[T(x) = t_0] = \sum_{x \in B_0} P[X = x]$$

$$= \sum_{x \in B_0} g(T(x), \theta) \cdot h(x).$$

Further

$$P[X=x / T(x)=t_0] = \frac{P[X=x,\ T(x)=t_0]}{P[T(x)=t_0]}$$

$$= \begin{cases} h(x) & , \text{if } T(x)=t_0 \\ 0 & , \text{if } T(x) \neq t_0 \end{cases}$$

Thus if $T(x)=t_0$ then

$$H(x) = \frac{g(t_0,\theta).\ h(x)}{g(t_0,\theta).\ \Sigma h(x)}$$

$$= \frac{h(x)}{\Sigma h(x)}.$$

Cramer-Rao inequality

If $T(x)$ is an unbiased estimator for $\psi(\theta)$, a function of parameter θ then the

$$var(T(x)) \geq \frac{\left[\frac{\partial}{\partial\theta}\psi(\theta)\right]^2}{E\left[\frac{\partial}{\partial\theta}\log L\right]^2} = \frac{[\psi'(\theta)]^2}{I(\theta)}.$$

Proof:- Let $X = x_1, x_2 --- x_n$ be a random variable drawn from $f(x,\theta)$ independently.

Let $L(x,\theta)$ the likelihood function of ~~x~~ X

using the regularity conditions. it is differentiable under integral sign.

$$\int \frac{\partial}{\partial\theta} L(x,\theta) = 0 \quad \text{i.e } E\left(\frac{\partial}{\partial\theta}\log L\right) = 0 \longrightarrow ①$$

we have $T(x)$ be an unbiased estimator for $\psi(\theta)$

Differentiating $\psi(\theta)$ w.r.t θ, we get

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$$\psi'(\theta) = \frac{\partial}{\partial\theta} E[T(x)] = \frac{\partial}{\partial\theta}\int T(x) . L(x,\theta)\, dx$$

$$= \int T(x) \left(\frac{\partial}{\partial\theta}\log L\right) dx$$

$$= E\left[T(x) \cdot \frac{\partial}{\partial\theta}\log L\right] \longrightarrow ②$$

$$\mathrm{cov}\left(T(x), \frac{\partial}{\partial\theta}\log L\right) = E\left(T(x)\cdot\frac{\partial}{\partial\theta}\log L\right) - E(T(x)) \cdot E\left(\frac{\partial}{\partial\theta}\log L\right).$$

$$= \psi'(\theta) - 0 \quad (\text{from } ①)$$

$$= \psi'(\theta) \longrightarrow ③$$

we have an inequality

$$\left[\mathrm{cov}\left(T(x), \frac{\partial}{\partial\theta}\log L\right)\right]^2 \leq \mathrm{var}(T(x)) \cdot \mathrm{var}\left(\frac{\partial}{\partial\theta}\log L\right)$$

$$\mathrm{var}(T(x)) \geq \frac{\left[\mathrm{cov}\left(T(x), \frac{\partial}{\partial\theta}\log L\right)\right]^2}{\mathrm{var}\left(\frac{\partial}{\partial\theta}\log L\right)}$$

$$\geq \frac{\left[\frac{\partial}{\partial\theta}\psi(\theta)\right]^2}{E\left[\frac{\partial}{\partial\theta}\log L\right]^2} = \frac{[\psi'(\theta)]^2}{I(\theta)}.$$

Regularity conditions (or) Applications

① The parameter space 'Ω' is non-degenerate open interval on real line ($-\infty$ to ∞)

② $\frac{\partial}{\partial\theta} L(x,\theta)$ is exists for all $x = (x_1, x_2 \ldots x_n)$, $\forall\ \theta \in \Omega$

③ The range of integration is independent of parameter θ, so that $f(x,\theta)$ is differentiable under integral sign.

i.e $\int \frac{\partial}{\partial\theta} L(x,\theta)$.

(4) The conditions of uniform convergence of integra... are satisfied so that differentiation under integral sign is valid.

(5) $I\theta = E\left[\frac{\partial}{\partial\theta} \log L\right]^2$ exists and is positive.

Lehaman-scheffee ~~Rao-blackwell~~ theorem

Let $T(x)$ be a statistic which is complete and sufficient for θ involved in $f(x,\theta)$, $\theta \in \Omega$. If there exists atleast one statistic $U(x) \in \mathcal{U}_\varphi$ then the

$\phi(T(x)) = E[U(x)/T(x)]$ is unique and uniformly MVUE for $\varphi(\theta)$.

Proof:- Assume that the class $\mathcal{U}_\varphi$ is non-degenerate so that these exists 2 elements $U_1(x)$ and $U_2(x)$ in $\mathcal{U}_\varphi$

Since $T(x)$ is sufficient for θ, by Rao-blackwell theorem.

$$\phi_j(T(x)) = E[U_j/T(x)] \in \mathcal{U}_\varphi, \quad j = 1, 2$$

Hence $E[\phi_1(T(x)) - \phi_2(T(x))] = 0, \ \forall \ \theta \in \Omega$.

Since $T(x)$ is complete sufficient statistic by def^n. of completeness. There exists a function

$$\eta[T(x)] = \phi_1(T(x)) - \phi_2(T(x))$$

$\Rightarrow E[\eta(T(x))] = 0$, for any $\theta \in \Omega$

$\Rightarrow \eta(t) = 0$ for each $T(x) = t$

ie $\phi_1(t) = \phi_2(t)$, for $t \in S^n$

Thus $U_1(x) = U_2(x) \in \mathcal{U}_\varphi$

and $\phi[T(x)] = E[U(x)/T(x)]$ is unique uniformly mini varience unbiased estimator for $\varphi(\theta)$.

Rao-Blackwell theorem

Let $T_n(x)$ be a sufficient statistic for $\psi(\theta)$ and $\mathcal{U}_\psi$ be a non empty class of unbiased estimators for $\psi(\theta)$. So that $U(x) \in \mathcal{U}_\psi$ and $\phi(t) = E[U(x)/T_n(x) = t]$ is a function of only 't' and independent of θ. then

(i) $\phi(T_n(x)) \in \mathcal{U}_\psi$

(ii) $var(\phi(T_n(x))) \leq var(U(x)),\ \forall\, \theta \in \Omega,\ U \in \mathcal{U}_\psi$.

proof:- Suppose that $\{f(x,\theta),\ \theta \in \Omega\}$ admits a sufficient Statistic $T_n(x)$ for $\psi(\theta)$.

Statistic $T_n(x)$ is sufficient.

$\Rightarrow U(x) \in \mathcal{U}_\psi$. since $T_n(x)$ is sufficient.

Given $\phi(t) = E[U(x)/T_n(x) = t]$

$$E[\phi(t)] = E[E[U(x)/T_n(x) = t]]$$
$$= E[U(x)]$$
$$= \psi(\theta).$$

which establishes the part 1 of the theorem.

In order to establish the part 2

consider $var[U(x)] = E[U(x) - E(U(x))]^2$

$$= E(U(x) - \psi(\theta))^2$$
$$= E[U(x) - \phi(t) + \phi(t) - \psi(\theta)]^2$$
$$= E[U(x) - \phi(t)]^2 + E[\phi(t) - \psi(\theta)]^2 - 2E(U(x) - \phi(t))[\phi(t) - \psi(\theta)]$$
$$= E[U(x) - \phi(t)]^2 + E(\phi(t) - \psi(\theta))^2$$
$$= E(U(x) - \phi(t)]^2 + E[\phi(t) - E(\phi(t))]^2$$
$$= E[U(x) - \phi(t)]^2 + V(\phi(t))$$

$\therefore var(U(x)) \geq var\, \phi(t)$.

Hence proved

Fisher's information on θ

Let $x_1, x_2 \ldots x_n$ with the random sample observed on the r.v's with pdf $f(x,\theta)$. Let $f(x,\theta)$ be differentiable w.r.t θ over measurable set c. The fishers information measured on θ contained in the r.v 'x' is defined as

$$I(\theta) = E\left[\frac{\partial}{\partial\theta} \log f(x,\theta)\right]^2$$

$$= var\left(\frac{\partial}{\partial\theta} \log f(x,\theta)\right) \quad \left(\because E\left[\frac{\partial \log f}{\partial\theta}\right] = 0\right).$$

Sufficiency of an estimator

A statistic $T(x)$ is said to be sufficient for θ. (or) the family of distributions $\{F_\theta ; \theta \in \Omega\}$ iff the conditional distribution of x given $T(x) = t$ is independent of θ.

$$f(x,\theta) = g(T(x),\theta)\, h(x).$$

Note:

① The sample observations themselves are always sufficient

② every statistic is not sufficient